Nucleon form factors with light Wilson quarks

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Introduction

High-precision study of excited states

Dirac and Pauli form factors near the physical $m_\pi$

Controlled study of finite-volume effects
Dirac and Pauli form factors:

\[
\langle p(P', s') | \bar{q} \gamma^\mu q | p(P, s) \rangle = \bar{u}(p', s') \left( \gamma^\mu F_1^q(Q^2) + i \sigma^{\mu\nu} \frac{\Delta_\nu}{2m_p} F_2^q(Q^2) \right) u(p, s),
\]

where \( \Delta = P' - P \), \( Q^2 = -\Delta^2 \).

- Isovector combination:

\[
F_{1,2}^v = F_{1,2}^u - F_{1,2}^d = F_{1,2}^p - F_{1,2}^n,
\]

where \( F_{1,2}^p,n \) are form factors of the electromagnetic current in a proton and in a neutron.

- Dirac and Pauli radii defined via slope at \( Q^2 = 0 \):

\[
F_{1,2}(Q^2) = F_{1,2}(0) \left( 1 - \frac{1}{6} r_{1,2}^2 Q^2 + O(Q^4) \right);
\]

\( F_2(0) = \kappa \), the anomalous magnetic moment.

- Proton charge radius, \( (r_E^2)^p = (r_1^2)^p + \frac{3\kappa_p}{2m_p^2} \), has 7\( \sigma \) discrepancy between measurements from \( e-p \) interactions and from Lamb shift in muonic hydrogen.
High-precision study of excited states

- USQCD ensemble with $N_f = 2 + 1$ Wilson-clover quarks coupled to stout-smeared gauge fields.
- $a \approx 0.114$ fm, $32^3 \times 96$, $m_\pi \approx 317$ MeV $\rightarrow m_\pi L = 5.9$
- Five source-sink separations: $T/a \in \{6, 8, 10, 12, 14\}$, $T \sim 0.7–1.6$ fm;
- Source tuned to optimize ground state overlap.
- $\sim 24000$ measurements yields reasonably precise results.
- Renormalization factors not yet computed, but this does not affect excited-states study.
Ground-state matrix elements from multiple $T$

- **Standard ratio-plateau method**: compute ratio

$$R(T, \tau) = \frac{C_{3pt}(T, \tau)}{C_{2pt}(T)} = c_{00} + c_{10}e^{-\Delta E}\tau + c_{01}e^{-\Delta E(T-\tau)} + c_{11}e^{-\Delta E T} + \ldots,$$

where $c_{00}$ is the desired ground-state matrix element. Then average a fixed number of points around $\tau = T/2$, yielding asymptotic errors that fall off as $e^{-\Delta E_{10} T/2}$.

- **Summation method** (PoS(Lattice 2010) 147 [1011.1358]; *ibid*. 303 [1011.4393]): compute sums

$$S(T) = \sum_{\tau} R(T, \tau) = b + c_{00} T + d T e^{-\Delta E T} + \ldots,$$

then find their slope, which gives $c_{00}$ with errors that fall off as $T e^{-\Delta E_{10} T}$.

- **Generalized pencil-of-function (GPoF) method** (AIP Conf. Proc. 1374, 621 [1010.0202]): recognize time-displaced operator $N^\tau(t) \equiv N(t + \tau)$ as linearly independent from $N(t)$. Use the variational method to find a linear combination of $N$ and $N^\tau$ that eliminates the first excited state. Applying the ratio-plateau method yields the ground-state with errors $e^{-\Delta E_{20} T/2}$.

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Isovector Dirac form factor $F_v^1(Q^2)$
Isovector Pauli form factor $F_2^v(Q^2)$

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BMW action and ensembles

- $N_f = 2 + 1$ tree-level clover-improved Wilson fermions coupled to double-HEX-smeared gauge fields.
- Pion mass ranging from 149 MeV to 356 MeV.
- Ten coarse lattices with $a = 0.116$ fm; one fine lattice with $a = 0.09$ fm.
- No disconnected diagrams, so we focus on isovector observables.
- Three source-sink separations for controlling excited-state contributions: $T \in \{0.9, 1.2, 1.4\}$ fm; use summation method for main results.
Areas of circles scale with number of measurements: largest is 24,000.
Chiral extrapolation

Use SU(2) heavy baryon ChPT, to order $\epsilon^3$ in SSE. Inputs:

- $F_\pi^0$, pion decay constant
- $\Delta$, delta-nucleon mass difference
- $g_A^0$, axial charge
- $c_A$, $\pi N\Delta$ coupling
- $c_V$, magnetic $\gamma N\Delta$ coupling

Fit parameters

- $(r_1^2)^{\nu}$: 1
- $\kappa^{\nu}$: 2
- $\kappa^{\nu}(r_2^2)^{\nu}$: 1
Dipole fitting to $F_1^\gamma(Q^2)$

$48^3 \times 48, m_\pi = 149$ MeV

$Q^2 < 0.5$ GeV$^2$
$Q^2 < 0.3$ GeV$^2$
$Q^2 < 0.2$ GeV$^2$
$Q^2 < 0.1$ GeV$^2$

$32^3 \times 48, m_\pi = 254$ MeV

$Q^2 < 0.7$ GeV$^2$
$Q^2 < 0.5$ GeV$^2$
$Q^2 < 0.3$ GeV$^2$

$24^3 \times 48, m_\pi = 254$ MeV

$Q^2 < 1.2$ GeV$^2$
$Q^2 < 0.7$ GeV$^2$
$Q^2 < 0.5$ GeV$^2$
Isovector Dirac radius \((r_1^2)^v\)

\[
(r_1^2)^v \text{ (fm}^2) = \frac{1}{r_2^2}v
\]

PDG

\[
\tilde{\mu}_p\coarse 48^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 96
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 24
\]

\[
\tilde{\mu}_p\coarse 24^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 24^3 \times 24
\]

\[
\tilde{\mu}_p\fine 32^3 \times 64
\]

\[
\tilde{\mu}_p\coarse 48^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 96
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 32^3 \times 24
\]

\[
\tilde{\mu}_p\coarse 24^3 \times 48
\]

\[
\tilde{\mu}_p\coarse 24^3 \times 24
\]

\[
\tilde{\mu}_p\fine 32^3 \times 64
\]

**ratio method: shortest \(T\)**

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Isovector Dirac radius $(r_1^2)^v$

\[ (r_1^2)^v \]

PDG

$\mu p$

coarse $48^3 \times 48$

coarse $32^3 \times 24$

coarse $32^3 \times 96$

coarse $32^3 \times 48$

coarse $32^3 \times 24$

fine $32^3 \times 64$

ratio method: middle $T$

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Isovector Dirac radius \((r_1^2)^v\)

\begin{align*}
\text{PDG} & \quad \text{coarse } 48^3 \times 48 \quad \text{coarse } 32^3 \times 24 \\
\mu p & \quad \text{coarse } 32^3 \times 96 \quad \text{coarse } 32^3 \times 48 \\
& \quad \text{coarse } 32^3 \times 24 \quad \text{coarse } 24^3 \times 48 \\
& \quad \text{coarse } 24^3 \times 24 \\
& \quad \text{fine } 32^3 \times 64
\end{align*}

ratio method: largest \(T\)
Isovector Dirac radius \((r_1^2)^v\)
Dipole fitting to $F_2^V(Q^2)$

$F_2^V(Q^2)$ for different $Q^2$ ranges:
- $Q^2 < 0.5$ GeV$^2$
- $Q^2 < 0.2$ GeV$^2$
- $Q^2 < 0.1$ GeV$^2$

$F_2^V(Q^2)$ for different $Q^2$ ranges:
- $Q^2 < 0.7$ GeV$^2$
- $Q^2 < 0.5$ GeV$^2$
- $Q^2 < 0.3$ GeV$^2$

$F_2^V(Q^2)$ for different $Q^2$ ranges:
- $Q^2 < 1.2$ GeV$^2$
- $Q^2 < 0.7$ GeV$^2$
- $Q^2 < 0.5$ GeV$^2$

$m_\pi = 149$ MeV

$m_\pi = 254$ MeV

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Dipole fitting to $F_2^v(Q^2)$

$F_2^v$ versus $Q^2$ for different Wilson quark masses:
- $48^3 \times 48, m_\pi = 149$ MeV:
  - $Q^2 < 0.5$ GeV$^2$
  - $Q^2 < 0.3$ GeV$^2$
  - $Q^2 < 0.2$ GeV$^2$
  - $Q^2 < 0.1$ GeV$^2$

- $32^3 \times 48, m_\pi = 254$ MeV:
  - $Q^2 < 0.7$ GeV$^2$
  - $Q^2 < 0.5$ GeV$^2$
  - $Q^2 < 0.3$ GeV$^2$

- $24^3 \times 48, m_\pi = 254$ MeV:
  - $Q^2 < 1.2$ GeV$^2$
  - $Q^2 < 0.7$ GeV$^2$
  - $Q^2 < 0.5$ GeV$^2$
Isovector anomalous magnetic moment $\kappa^v$

<table>
<thead>
<tr>
<th>PDG</th>
<th>coarse $32^3 \times 24$</th>
</tr>
</thead>
<tbody>
<tr>
<td>coarse $48^3 \times 48$</td>
<td>coarse $24^3 \times 48$</td>
</tr>
<tr>
<td>coarse $32^3 \times 96$</td>
<td>coarse $24^3 \times 24$</td>
</tr>
<tr>
<td>coarse $32^3 \times 48$</td>
<td>fine $32^3 \times 64$</td>
</tr>
</tbody>
</table>

$\kappa^v_{\text{norm}}$ vs $m_\pi$ (GeV)

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Isovector Pauli radius \((r_2^2)^v\)

![Graph showing \(\kappa_{\text{norm}}(r_2^2)^v\) vs. \(m_\pi\) (GeV) for different quark masses.](image)

- **DA**: coarse \(48^3 \times 48\), coarse \(32^3 \times 24\)
- **PDG**: coarse \(32^3 \times 96\), coarse \(24^3 \times 48\)
- **Fine**: coarse \(32^3 \times 64\)

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Sachs form factors

\[ G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{2m_N} F_2(Q^2) \]
\[ G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \]

- Slopes at \( Q^2 = 0 \) give rms charge and magnetic radii.
- Compare:
  1. Isovector \( G_E(Q^2) \), \( G_M(Q^2) \) from lattice calculation.
  2. Parameterization of experimental data:
     4 parameters for each of \( G_{Ep} \), \( G_{Mp} \), \( G_{Mn} \); 2 parameters for \( G_{En} \): determined from fit to experiment.
Electric form factor $G_E^v(Q^2)$

$m_\pi = 149$ MeV, ratio method, $T = 10a$
Electric form factor $G^v_E(Q^2)$

$m_\pi = 149$ MeV, summation method; $p = 0.64$
Electric form factor $G_E^v(Q^2)$

$m_\pi = 254 \text{ MeV}, \text{ summation method}; p = 3 \times 10^{-5}$
Magnetic form factor $G_M^V(Q^2)$

$m_\pi = 149 \text{ MeV}, \text{ ratio method, } T = 10a; p = 0.0006$
Magnetic form factor $G_M^V(Q^2)$

$m_\pi = 149 \text{ MeV}$, summation method; $p = 0.81$
Magnetic form factor $G_M^V(Q^2)$

$m_\pi = 254$ MeV, summation method; $p = 0.0007$
Controlled study of finite $L_s$ and $L_t$ effects

- Four lattice ensembles at $m_\pi \approx 250$ MeV with $a = 0.116$ fm that differ only in their volume: $32^3 \times 48$, $24^3 \times 48$, $32^3 \times 24$, and $24^3 \times 24$.

- $m_\pi L_s = 3.6$ and 4.8
- $m_\pi L_t = 3.6$ and 7.2
- Test “$m_\pi L = 4$” rule of thumb.

Results:
  - Effects consistent with zero when comparing noisy summation data.
  - Shortest source-sink separation data suggest at $m_\pi L_s = 4$:
    - possible $\sim -5\%$ effect on $\kappa^\nu$ and $(r_2^\nu)^\nu$
    - effect on $(r_1^\nu)^\nu$ is consistent with zero and less than 2%.
  - No effect seen for $g_A$. 
Summary

- With both
  1. near-physical $m_\pi$
  2. reduced excited-state contributions
good agreement is achieved with experiment for isovector vector form factors.

- High-precision calculations at $m_\pi \approx 317$ MeV corroborate the excited-state behavior seen in the noisier calculations at near-physical pion masses.

- Dedicated finite-$L_s$ and $L_t$ study at $m_\pi \approx 250$ MeV finds that $m_\pi L = 4$, effects are generally consistent with zero.
Source tuning, high-precision ensemble

Wuppertal smearing with $\alpha = 3$, $N = 35$ using APE-smeared ($A = 2.85$, $N = 25$) gauge links.
Axial and induced pseudoscalar form factors:

\[
\langle p', \lambda' | \bar{q} \gamma^\mu \gamma_5 q | p, \lambda \rangle = \bar{u}(p', \lambda') \left( \gamma^\mu G_A^q(Q^2) + \frac{\Delta^\mu}{2m} G_P^q(Q^2) \right) \gamma_5 u(p, \lambda). 
\]

Generalized form factors of the quark energy-momentum operator

\[
\langle p', \lambda' | \bar{q} \gamma^{\{\mu} i \overset{\leftrightarrow}{D}^{\nu\}} q | p, \lambda \rangle = \bar{u}(p', \lambda') \left( \bar{p}^{\{\mu} \gamma^{\nu\}} A_{20}^q(Q^2) + \frac{i\bar{p}^{\{\mu} \sigma^{\mu\}}_\alpha \Delta_\alpha}{2m} B_{20}^q(Q^2) + \frac{\Delta^{\{\mu} \Delta^{\nu\}}}{m} C_2^q(Q^2) \right) u(p, \lambda) 
\]
Source and sink momenta

| #  | \(\langle \vec{n}'|\vec{n}\rangle\)       |
|----|---------------------------------------|
| 0  | \(\langle 0, 0, 0|0, 0, 0\rangle, \langle -1, 0, 0|-1, 0, 0\rangle\) |
| 1  | \(\langle 0, 0, 0|1, 0, 0\rangle\)           |
| 2  | \(\langle -1, 0, 0|-1, 1, 0\rangle\)         |
| 3  | \(\langle 0, 0, 0|1, 1, 0\rangle\)           |
| 4  | \(\langle -1, 0, 0|-1, 1, 1\rangle\)         |
| 5  | \(\langle -1, 0, 0|0, 1, 0\rangle\)           |
| 6  | \(\langle 0, 0, 0|1, 1, 1\rangle\)           |
| 7  | \(\langle -1, 0, 0|0, 1, 1\rangle\)         |
| 8  | \(\langle -1, 0, 0|0, 1, 1\rangle\)         |
| 9  | \(\langle -1, 0, 0|1, 1, 0\rangle\)           |
| 10 | \(\langle -1, 0, 0|1, 1, 1\rangle\)           |

\[\vec{p} = \frac{2\pi}{L_s}\vec{n}\]
Isovector Axial form factor $G_A^v(Q^2)$

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Isovector induced pseudoscalar form factor $G_P^v(Q^2)$

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Isovector generalized form factor $A^{v}_{20}(Q^2)$

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Isovector anomalous magnetic moment $\kappa^v$

$T = 8$ summation
$T = 10$ GPoF
$T = 12$

$\kappa^v_{\text{norm}}$

$32^3 \times 48$
$32^3 \times 24$
$24^3 \times 48$
$24^3 \times 24$
Isovector Pauli radius \((r_2^2)^v\)

![Graph showing isovector Pauli radius](image)

- \(T = 8\) summation
- \(T = 10\) G PoF
- \(T = 12\) G PoF

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Axial charge $g_A$

\begin{align*}
T = 8 & \quad \text{summation} \\
T = 10 & \quad \text{GPoF} \\
T = 12 & 
\end{align*}

\begin{align*}
32^3 \times 48 \\
32^3 \times 24 \\
24^3 \times 48 \\
24^3 \times 24
\end{align*}
Isovector quark momentum fraction $\langle x \rangle_{u-d}$

![Graph showing the isovector quark momentum fraction for different lattice sizes and temperatures.](image-url)
\( L_s \to \infty, L_t \to \infty \) extrapolation

Fit \( A + Be^{-m_\pi L_s} + Ce^{-m_\pi L_t} \) to summation data:

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( Be^{-4} )</th>
<th>( Ce^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (r_1^2) \nu )</td>
<td>0.338(48)</td>
<td>0.008(38)</td>
<td>0.003(28)</td>
</tr>
<tr>
<td>( \kappa \nu )</td>
<td>3.19(36)</td>
<td>-0.12(35)</td>
<td>-0.20(23)</td>
</tr>
<tr>
<td>( (r_2^2) \nu )</td>
<td>0.476(98)</td>
<td>-0.032(106)</td>
<td>-0.028(68)</td>
</tr>
<tr>
<td>( g_A )</td>
<td>1.204(67)</td>
<td>-0.009(54)</td>
<td>-0.016(39)</td>
</tr>
<tr>
<td>( \langle x \rangle_{u-d} )</td>
<td>0.178(18)</td>
<td>0.021(14)</td>
<td>-0.014(10)</td>
</tr>
</tbody>
</table>
Relative error, $m_\pi L_s = 4$

![Graph showing relative error with $m_\pi L_s = 4$ and T = 8, T = 10, T = 12.](image)

Plotted: $e^{-4}B/A$
Relative error, $m_\pi L_t = 4$

Plotted: $e^{-4}C/A$