Probing the nucleon and its excitations in Full QCD

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August 1st, 2013

Outline

Motivation

- A long standing issue within the lattice community has been the systematically low value for g_A
- A number of systematic effects have been proposed as the cause – many of these were detailed in the Hadron Structure talk at last years Lattice Conference[1]
- The CSSM and others have begun utilising correlation matrix techniques to access excited state properties and transitions[2–7]
- Correlation matrix techniques provide us with a systematic framework to examine excited state effects on the calculation of g_A

[1] *H. W. Lin, PoS LATTICE* ***2012****, 013 (2012)*

[2] *B. J. Owen et al., Phys. Lett. B* ***723****, 217 (2013) [arXiv:1212.4668 [hep-lat]]*
[3] *B. J. Owen et al., PoS LATTICE* ***2012****, 173 (2012)*
[4] *B. J. Menadue et al., PoS LATTICE* ***2012****, 178 (2012)*
[5] *J. Dudek et al., Phys. Rev. D* ***79****, 094504 (2009) [arXiv:0902.2241 [hep-ph]].*
[6] *J. Bulava et al., JHEP* ***1201****, 140 (2012) [arXiv:1108.3774 [hep-lat]]*
[7] *T. Maureret al., arXiv:1202.2834 [hep-lat] (2012)*

CM Analysis

- A systematic framework for generating ideal operators for Hamiltonian Eigenstates
- Require a basis of operators: $\{\chi_i\}\,;\ i \in [1, N]$
- Calculate set of cross-correlation functions

$$
\begin{aligned}
\mathcal{G}_{ij}(t, \vec{p}; \Gamma) &= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \mathrm{tr}(\Gamma \langle \Omega | \chi_i(x) \bar{\chi}_j(0) | \Omega \rangle) \\
&= \sum_{\alpha=0}^{N-1} e^{-E_\alpha(\vec{p})\, t} Z_i^\alpha(\vec{p}) \bar{Z}_j^\alpha(\vec{p}) \mathrm{tr}\left(\frac{\Gamma(\not{p} + m_\alpha)}{2E_\alpha(\vec{p})} \right)
\end{aligned}
$$

where Z_i^α, $\bar{Z}_j^\alpha$ are the couplings of sink operator (χ_i) and source operator $(\bar{\chi}_j)$ for the state α

CM Analysis (cont)

- Desire N linearly independent sink (ϕ_α) and source ($\bar{\phi}_\alpha$) operators
- Ideally, we want these operators to satisfy

$$\langle\, \Omega \,|\, \phi^\beta \,|\, m_\alpha, p, s \,\rangle = \delta_{\alpha\beta} \mathcal{Z}^\alpha(\vec{p}) \sqrt{\frac{m_\alpha}{E_\alpha(\vec{p})}} u(p,s)$$

- use our basis of operators to construct these new operators

$$\left.\begin{aligned} \bar{\phi}_\alpha(x,\vec{p}) &= \sum_{i=1}^{N} u_i^\alpha(\vec{p}) \bar{\chi}_i(x) \\ \phi_\alpha(x,\vec{p}) &= \sum_{i=1}^{N} v_i^\alpha(\vec{p}) \chi_i(x) \end{aligned}\right\} \quad \text{optimal coupling to state } |\, m_\alpha, p, s \,\rangle$$

CM Analysis (cont)

- Using the above definitions, it is easy to show that the desired values for u_i^α, v_i^α are the components of the eigenvectors for the following eigenvector equations

CM Eigenvalue Equation

$$[\,\mathcal{G}^{-1}(t_0,\vec{p};\Gamma)\,\mathcal{G}(t_0+\Delta t,\vec{p};\Gamma)\,]_{ij}u_j^\alpha(\vec{p}) = \lambda^\alpha u_j^\alpha(\vec{p}) \tag{1a}$$

$$v_i^\alpha(\vec{p})[\,\mathcal{G}(t_0+\Delta t,\vec{p};\Gamma)\,\mathcal{G}^{-1}(t_0,\vec{p};\Gamma)\,]_{ij} = \lambda^\alpha v_i^\alpha(\vec{p}) \tag{1b}$$

where $\lambda^\alpha = e^{-E_\alpha(\vec{p})\Delta t}$.

- Using $v_i^\alpha(\vec{p})$, $u_j^\alpha(\vec{p})$ we are able to project out the correlation function for the state $|\,m_\alpha, p, s\,\rangle$

$$\mathcal{G}_\alpha(t,\vec{p};\Gamma) = v_i^\alpha(\vec{p})\mathcal{G}_{ij}(t,\vec{p};\Gamma)u_j^\alpha(\vec{p})$$

Our operator basis

- For the best results, one needs a basis of operators which are significantly independent to ensure that their overlap with energy eigenstates is different.
- Smearing alters the overlap of an operator with the eigenstates
- In this work we will use Gauge Invariant Gaussian smearing as a method to increase the number of available operators

Gauge Invariant Gaussian smearing

- Gauge invariant smeared sources are constructed via an iterative process evaluated on point source $\psi_0(x)$,

$$\psi_i(x) = \sum_{x'} F(x, x')\psi_{i-1}(x')$$

where

$$F(x, x') = (1 - \alpha)\delta_{x',x} + \frac{\alpha}{6}\sum_{\mu=1}^{3}[U_\mu(x)\delta_{x',x+\hat{\mu}} + U_\mu^\dagger(x - \hat{\mu})\delta_{x',x-\hat{\mu}}].$$

- Thus we smeared source with n sweeps of smearing is

$$\psi_n(x) = \sum_{x'} F^n(x, x')\psi_0(x')$$

Smearing Parameters

In this work we have used four levels of smearing with smearing fraction $\alpha = 0.7$. This choice was determined[1] to best span the operator space resulting in the best projection of energy eigenstates.
Below we list the corresponding rms-radii for these sources.

Table : The rms radii for the various levels of smearing considered in this work.

Sweeps of smearing	rms radius (fm)
16	0.216
35	0.319
100	0.539
200	0.778

[1] *M. S. Mahbub et al., Phys. Lett. B.* ***707****, (2012) 389*

CM Analysis for 3pt-functions

- As we have for two-point case, we calculate the set of three-point cross-correlation functions,

$$G_{ij}(\vec{p}\,', \vec{p};\, t_2, t_1;\, \Gamma') = \sum_{\vec{x}_1, \vec{x}_2} e^{-i\vec{p}\,'\cdot\vec{x}_2}\, e^{+i(\vec{p}\,'-\vec{p})\cdot\vec{x}_1} \operatorname{tr}\left(\Gamma' \langle \Omega | \chi_i(x_2) \mathcal{O}(x_1) \bar{\chi}_j(0) | \Omega \rangle \right),$$

  where $\mathcal{O}(x)$ is the current operator to be inserted.
- This we can expand in an analogous manner to give couplings, exponential factors and the desired matrix element,

$$G_{ij}(\vec{p}\,', \vec{p};\, t_2, t_1;\, \Gamma') = \sum_{\alpha,\,\beta} e^{-E_\beta(\vec{p}\,')(t_2-t_1)}\, e^{-E_\alpha(\vec{p})t_1} Z_i^\beta(\vec{p}\,')\, \bar{Z}_j^\alpha(\vec{p}) \sqrt{\frac{m_\alpha\, m_\beta}{E_\alpha(\vec{p})\, E_\beta(\vec{p}\,')}} \operatorname{tr}\left(\Gamma' \sum_{s',s} u(p', s') \langle\, \beta, p', s' \,|\, \mathcal{O}(0) \,|\, \alpha, p, s \,\rangle\, \bar{u}(p, s) \right).$$

CM Analysis for 3pt-functions (cont)

- The eigenvectors derived from the two-point analysis can be used to project out the three-point function
- The key is to ensure that the eigenvector corresponds to the momentum to be projected at the source / sink

$$G^{\alpha}(\vec{p}\,', \vec{p};\, t_2, t_1;\, \Gamma') = v_i^{\alpha}(\vec{p}\,')\, G_{ij}(\vec{p}\,', \vec{p};\, t_2, t_1;\, \Gamma')\, u_j^{\alpha}(\vec{p})\,.$$

- With the desired state now isolated, one simply uses the projected correlation functions in the ratio to extract the matrix element.
- In this work we have used the following ratio,

$$R^{\alpha}(\vec{p}\,', \vec{p};\, \Gamma', \Gamma) = \sqrt{\frac{G^{\alpha}(\vec{p}\,', \vec{p};\, t_2, t_1;\, \Gamma')\, G^{\alpha}(\vec{p}, \vec{p}\,';\, t_2, t_1;\, \Gamma')}{G^{\alpha}(\vec{p}, t_2;\, \Gamma)\, G^{\alpha}(\vec{p}\,', t_2;\, \Gamma)}}\,.$$

Accessing the Nucleon Axial Charge

- Probe the nucleon via the axial current.
  This vertex can be expressed as,

$$\langle\, p(p', s') \,|\, A_\mu^{u-d} \,|\, p(p, s) \,\rangle = \left(\frac{m^2}{E_{p'} E_p} \right)^{1/2}$$
$$\bar{u}(p', s') \left[\gamma_\mu \gamma_5 G_A(Q^2) + \gamma_5 \frac{q_\mu}{2m} G_P(Q^2) \right] u(p, s) \,,$$

  where $A_\mu^{u-d} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$.

- Selecting the nucleon rest frame, we extract correlation matrix improved value for g_A from the following ratio

$$g_A^{CM} = \frac{v_i^0(\vec{0}\,)\,\mathcal{G}_{ij}(\vec{0}\,,\vec{0}\,;\,t_2,t_1;\,\Gamma_3)\,u_j^0(\vec{0}\,)}{v_i^0(\vec{0}\,)\,\mathcal{G}_{ij}(\vec{0}\,,t;\,\Gamma_4)\,u_j^0(\vec{0}\,)}\,.$$

Configuration Details

- For this calculation we are working with the PACS-CS (2+1)-flavour Full QCD ensembles[1] made available through the ILDG
- These are $32^3 \times 64$ lattices with $\beta = 1.9$, corresponding to a physical lattice spacing of $0.0907(13)\ \mathrm{fm}$
- Iwasaki gauge action and pre-conditioned Wilson-Clover quark action
- There are five light quark-masses resulting in pion masses that range from $622\ \mathrm{MeV}$ through to $156\ \mathrm{MeV}$
- Here we focus on the second lightest mass with $m_\pi^2 = 290$ MeV over 200 gauge field configurations

[1] *S. Aoki et al., Phys. Rev. D* **79**, *034503 (2009)*

Simulation Parameters

- The variational parameters $t_0 = 18$ and $\Delta t = 2$ were used having been shown[1] to give the best balance between systematic and statistical uncertainties
- For this calculation, we choose to use fixed current SST method where we use a naive axial current inserted at $t_C = 21$ relative to the source at $t_0 = 16$
- The current renormalization parameter $Z_A = 0.781(20)$ and was determined[2] non-perturbatively using a Schrödinger functional scheme

[1] *M. S. Mahbub et al., Phys. Lett. B. 707, (2012) 389*

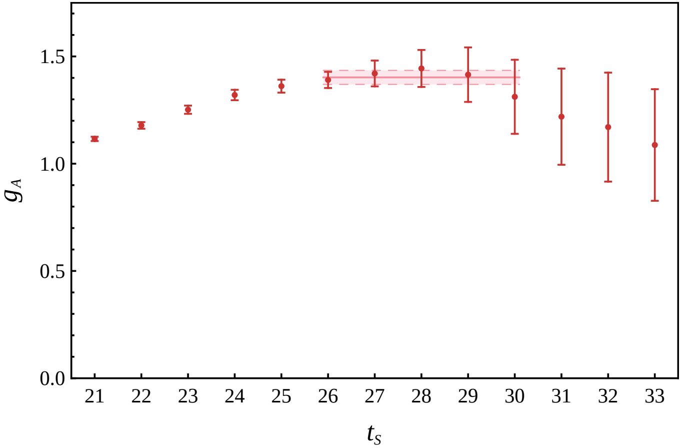
[2] *S. Aoki et al., JHEP. 1008 (2010) 101*

Comparison between methods

- We shall compare standard single correlation function method with our correlation matrix improved method.
- Red dataset will be smeared source to point sink
- Purple dataset will be smeared source to smeared sink
- Blue dataset will be correlation matrix method
- The fits have been chosen by criterion that the χ^2_{dof} obtained via a covariance matrix analysis be as close to 1 as possible

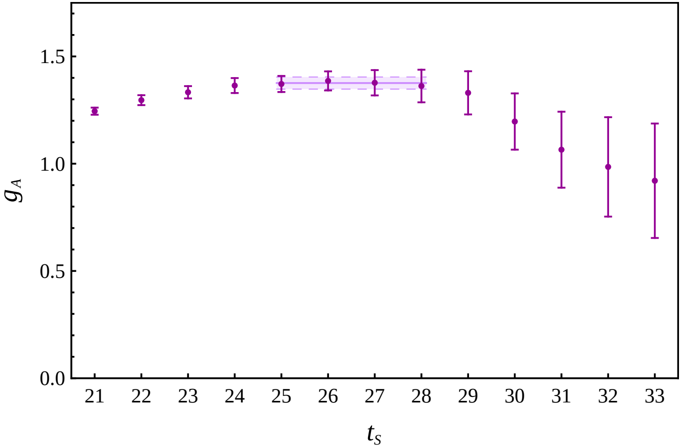
Standard method: smeared source to point sink

- Search for plateau following current at $t_C = 21$
- Clear presence of excited state effects from times 21 to 25
- Based on the χ^2_{dof}, earliest possible fit-window is at $t_S = 26$



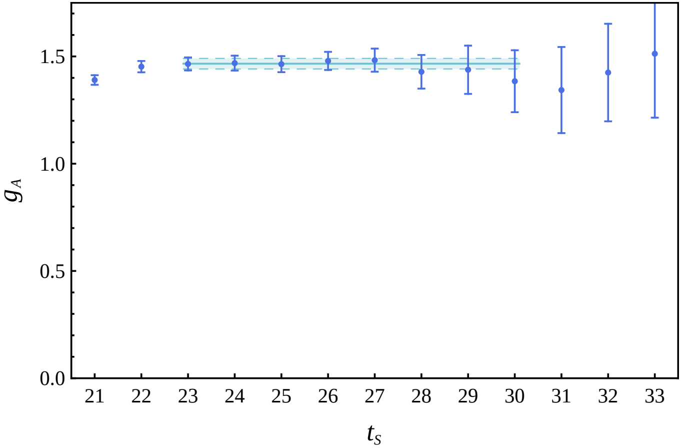
Standard method: smeared source to smeared sink

- Diminished excited state effects, but still present
- Earliest possible fit-window is now at $t_S = 25$

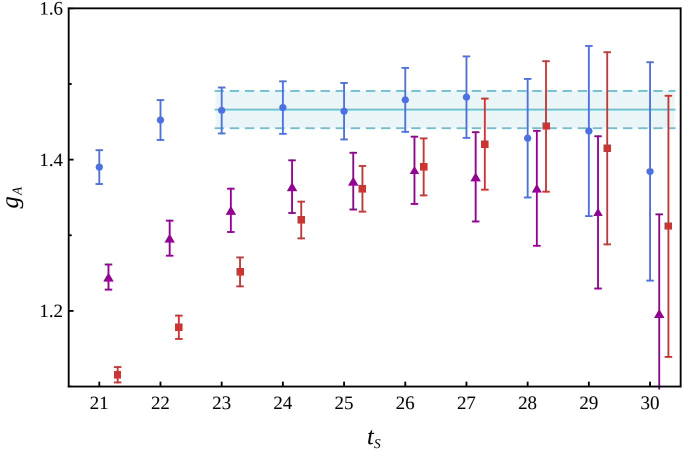


Variational method

- Excited state effects limited to a single time-slice after the current
- Early onset of ground state dominance provides smaller statistical uncertainties



Overlay of previous three plots



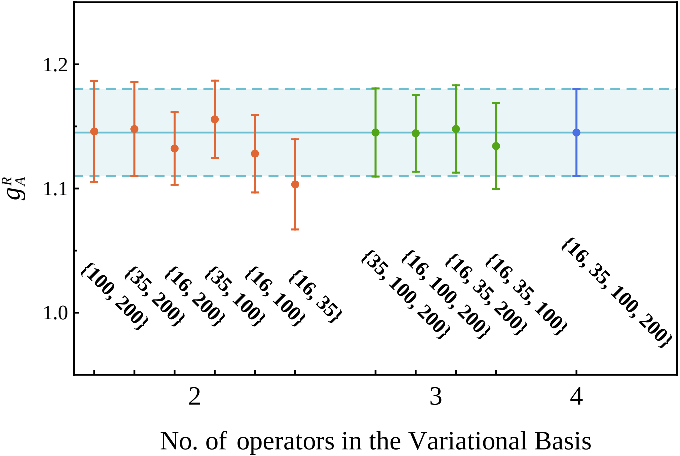
- Note that the smeared-smeared approach has no overlap with the fit for the correlation matrix approach
- Failure to obtain ground state dominance by $t_C = 21$ leads to incorrect plateau values in the standard method

Fine tuning

- In practice one could tune their smearing to give the best overlap with the ground state
- This can be an expensive exercise in itself and this tuning is unique to a single state at a particular quark mass
- Using variational approach as we have here automatically gives optimal overlap without the need for tedious fine tuning
- In practice, if one wants to control excited state contaminations, the only robust solution is to remove, not suppress these contributions

Examining the operator basis

- The larger the variational basis the more excited states are removed
- Here we have considered all subsets of our variational basis to consider how small one can make the operator basis



Cost vs Benefit

- A concern with a correlation matrix approach is the associated cost
- In our implementation (fixed current SST), we require $2n$ inversions per configuration where n is the number of smearings
  - A total of 8 inversions per configuration were used herein
  - The minimal method requires 2 inversions per configuration
  - Using a fixed sink SST inversion would require an inversion for each source-sink smearing combination resulting in $n^2 + n$ inversions per configurations
- At large Euclidean times, the standard approach is consistent with the correlation matrix approach, albeit with larger errors
- It is worth considering what the required increase in statistical sample would be for the standard method to produce similar errors

Cost vs Benefit (cont)

- Consider the ratio

$$\frac{N_{\text{required}}}{N_{\text{current}}} = \left(\frac{(\Delta g_A)_{\text{current}}}{(\Delta g_A)_{\text{desired}}}\right)^2 = \left(\frac{(\Delta g_A)_{\text{sm-sm}}}{(\Delta g_A)_{\text{CM}}}\right)^2 .$$

- As the leading error dominates the error on the fit, we will use this as being indicative of the uncertainty in g_A
- For the variational method
  - Ground dominance of the two-point function occurs at $t = 21$
  - Ground dominance of the three-point function occurs two time slices after the current
  - At $t_S = 21 + 2 = 23$ we have $(\Delta g_A)_{\text{CM}} = 0.030$
- For the standard method using smeared source and smeared sink
  - Ground dominance of the two-point function occurs at $t = 23$
  - Ground dominance of the three-point function (from Fig. 2) occurs six time slices after the current
  - At $t_S = 23 + 6 = 29$ we have $(\Delta g_A)_{\text{sm-sm}} = 0.101$

Cost vs Benefit (cont)

- Using these values we get

$$\frac{N_{\text{required}}}{N_{\text{current}}} = \left(\frac{0.101}{0.030}\right)^2 \simeq 11.3.$$

  a factor of 11 increase in statistics!

- If one were solely interested in properties of a single state, ie. ground state, then one could use the optimised sources generated via the two-point correlation matrix as input for the SST inversion
- This reduces the cost from $2n$ (fixed current) or $n^2 + n$ (fixed sink) down to $n + 1$ for either case
- Based on Fig. 5, a basis of 3 (well chosen) operators is enough to isolate the ground state – this gives a total cost of 4 inversions per configuration
- Thus for a factor of two increase in cost, we effectively obtain an order of magnitude improvement in the statistics

Concluding Remarks

- As the variational approach enables one to:
  - rapidly isolate the ground state following the source enabling an earlier current insertion, and
  - rapidly isolate the ground state again after inserting the current enabling an earlier Euclidean time fit,
- The associated reduction in the error bar from this process outweighs the increased cost in constructing the matrix of cross-correlators
- In practice, if one wants to control excited state contaminations, the only robust solution is to remove, not suppress these contributions
- The variational approach offers improved access to observables and will be instrumental in the era of precision matrix element determinations