Determination of the $A_2$ amplitude of $K \rightarrow \pi \pi$ decays

Tadeusz Janowski

RBC-UKQCD collaboration
University of Southampton

tj1g11@soton.ac.uk

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RBC+UKQCD Collaboration

RBC
Ziyuan Bai, Thomas Blum, Norman Christ, Tomomi Ishikawa, Taku Izubuchi, Luchang Jin, Chulwoo Jung, Taichi Kawanai, Chris Kelly, Hyung-Jin Kim, Christoph Lehner, Jasper Lin, Meifeng Lin, Robert Mawhinney, Greg McGlynn, David Murphy, Shigemi Ohta, Eigo Shintani, Amarjit Soni, Oliver Witzel, Hantao Yin, Jianglei Yu, Daiqian Zhang

UKQCD
Rudy Arthur, Peter Boyle, Hei-Man Choi, Luigi Del Debbio, Shane Drury, Jonathan Flynn, Julien Frison, Nicolas Garron, Jamie Hudspith, Tadeusz Janowski, Andreas Jüttner, Richard Kenway, Andrew Lytle, Marina Marinkovic, Brian Pendleton, Antonin Portelli, Enrico Rinaldi, Chris Sachrajda, Ben Samways, Karthee Sivalingam, Matthew Spraggs, Tobi Tsang
Introduction

In the isospin limit one can classify two kaon decay channels:

- $K \rightarrow (\pi\pi)_{I=2}$
- $K \rightarrow (\pi\pi)_{I=0}$

An experimental observation shows that the ratio of real parts of amplitudes for these decays is:

$$\frac{\text{Re}(A_0)}{\text{Re}(A_2)} \approx 22.5$$

This is known as ‘$\Delta I = 1/2$ rule’. In our recent work, we calculated the $A_2$ amplitude at physical kinematics (arXiv:1206.5142). This calculation was done using a single ensemble, which led to uncontrolled systematic errors related to cutoff effects.
# Lattice parameters

Two new ensembles:

<table>
<thead>
<tr>
<th></th>
<th>$48^3 \times 96$</th>
<th>$64^3 \times 128$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size [fm]</td>
<td>5.49</td>
<td>5.48</td>
</tr>
<tr>
<td>Gauge action</td>
<td>Iwasaki</td>
<td>Iwasaki</td>
</tr>
<tr>
<td>Fermion action</td>
<td>DWF</td>
<td>DWF</td>
</tr>
<tr>
<td>$L_s$</td>
<td>24</td>
<td>10</td>
</tr>
<tr>
<td>$M$</td>
<td>1.8</td>
<td>1.8</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2.13</td>
<td>2.25</td>
</tr>
<tr>
<td>$a^{-1}$ [GeV]</td>
<td>1.73(1)</td>
<td>2.30(4)</td>
</tr>
<tr>
<td>$am_{ud}$</td>
<td>0.00078</td>
<td>0.000678</td>
</tr>
<tr>
<td>$am_s$</td>
<td>0.0362</td>
<td>0.02661</td>
</tr>
<tr>
<td>$am_{res}$</td>
<td>$6.19(6) \times 10^{-4}$</td>
<td>$2.93(8) \times 10^{-4}$</td>
</tr>
<tr>
<td>Pion mass [MeV]</td>
<td>139</td>
<td>135</td>
</tr>
<tr>
<td>Kaon mass [MeV]</td>
<td>499</td>
<td>495.7</td>
</tr>
<tr>
<td>Number of configurations</td>
<td>44</td>
<td>21</td>
</tr>
</tbody>
</table>

We can get away with small number of configurations thanks to AMA procedure: arXiv:1208.4349
Operator product expansion

Weak Hamiltonian can be expanded using operator product expansion:

\[ H_{\text{W}} = \frac{G_F}{\sqrt{2}} V_{ud}^* V_{us} a^3 \sum_i C_i Q_i \]

Then the amplitudes can be written as:

\[ A_{2/0} = F \langle (\pi \pi)_{I=2} | H_{\text{W}} | K \rangle \]

We are therefore interested in the following 3-point functions:

\[ M_i^{I=2/0} \equiv \langle (\pi \pi)_{I=2} | Q_i^{\Delta I=(3/2)/(1/2)} | K \rangle \]

Only 3 operators contribute to \( K \to (\pi \pi)_{I=2} \) process. We label them according to their chiral transformation properties \((27,1), (8,8), (8,8)_{mx}\):

- \((27,1)\) is the dominant contribution to \(\text{Re}(A_2)\)
- \((8,8)\) operators are the dominant contributions to \(\text{Im}(A_2)\)
Renormalization

Wilson coefficients have been calculated in $\overline{MS}$ scheme, so we need to express our lattice results in $\overline{MS}$ scheme as well. This requires the use of an intermediate scheme, like RI-SMOM

\[ M_i^{\text{LAT}} \rightarrow M_i^{\text{RI-SMOM}} \rightarrow M_i^{\overline{MS}} \]

Only operators in the same $SU(3)_L \times SU(3)_R$ operators mix under renormalization.

\[ p^2 = q^2 = (p - q)^2 = \mu^2 \]
Two pion momentum

In centre of mass frame, the ground state for the two pion system will correspond to each pion being at rest. To avoid this problem we use antiperiodic boundary conditions for the d quark (and periodic for the u quark). The allowed momenta for the $\pi^+$ meson become:

$$p = \pm \frac{\pi}{L}, \pm \frac{3\pi}{L}, \ldots$$

In $I=2$ case only, we can use the Wigner-Eckart theorem:

$$\sqrt{2} \langle \pi^+\pi^0 | \langle (\pi\pi)_{I^3=1} | Q^{\Delta I=3/2} | K^+ \rangle = \sqrt{3} \langle (\pi\pi)_{I^3=2} | Q^{\Delta I=3/2} | K^+ \rangle$$

Both $48^3$ and $64^3$ ensembles tuned so that antiperiodic boundary conditions in 3 directions (which induce momentum $p = \frac{\sqrt{3}\pi}{L}$) correspond to physical kinematics.
Finite volume effects

Need to take into account interactions in the final state.

\[ F^2 = 4\pi \left( q \frac{\partial \phi}{\partial q} + p \frac{\partial \delta}{\partial p} \right) \frac{m_K E_{\pi\pi}}{p^3} \]

with:

\[ \tan \phi = \frac{q\pi^{3/2}}{Z_{00}(1; q)} \]

ulist:
- \( \delta \) is the 2-pion s-wave phase shift
- \( \delta \) can be computed from the lattice using Lüschner quantization condition, but...
- \( \frac{\partial \delta(p)}{\partial p} \) can not, so we have to approximate
$48^3 \, K \rightarrow \pi\pi$ 3-point correlation functions

Kaon - 2 pion separation 26

$C_i^{K \rightarrow \pi\pi}(t) = N_{\pi\pi} N_K M_i e^{-(m_K - E_{\pi\pi})t_{\text{top}}} e^{-E_{\pi\pi} t_{\pi\pi}}$
64^3 \ K \to \pi\pi \text{ 3-point correlation functions}

Kaon - 2 pion separation 26
\[ C^K_{\pi\pi}(t) = N_{\pi\pi} N_K M_i e^{-(m_K-E_{\pi\pi}) t_{\text{top}}} \ e^{-E_{\pi\pi} t_{\pi\pi}} \]
## Summary of results (PRELIMINARY)

<table>
<thead>
<tr>
<th></th>
<th>48$^3$</th>
<th>64$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dispersion relation ($c^2$)</td>
<td>0.999(9)</td>
<td>1.008(10)</td>
</tr>
<tr>
<td>Pion mass [MeV]</td>
<td>139.4(3)</td>
<td>136.0(3)</td>
</tr>
<tr>
<td>Kaon mass [MeV]</td>
<td>498.9(4)</td>
<td>495.6(5)</td>
</tr>
<tr>
<td>$(\pi\pi)_{I2}$ energy [MeV]</td>
<td>497.8(43)</td>
<td>503.7(38)</td>
</tr>
<tr>
<td>Re($A_2$)[GeV]</td>
<td>$1.368(41) \times 10^{-8}$</td>
<td>$1.358(28) \times 10^{-8}$</td>
</tr>
<tr>
<td>Im($A_2$)[GeV]</td>
<td>$-6.30(12) \times 10^{-13}$</td>
<td>$-6.31(10) \times 10^{-13}$</td>
</tr>
</tbody>
</table>

C.f. 32$^3$ result:

Re($A_2$) = $1.381(41) \times 10^{-8}$ GeV  
Im($A_2$) = $-6.54(46) \times 10^{-13}$ GeV
Continuum extrapolation for $A_2$ amplitude (PRELIMINARY)

Values shown are the amplitudes in $\overline{MS}$ scheme at 3GeV

$\text{Re}(A_2) = 1.345(84) \times 10^{-8}$ GeV

$\text{Im}(A_2) = -6.32(28) \times 10^{-13}$ GeV
## Error budget (VERY PRELIMINARY)

<table>
<thead>
<tr>
<th>Source</th>
<th>Re($A_2$)</th>
<th>Im($A_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice artefacts</td>
<td>15% $\downarrow$ 6%^{+\text{stat}}</td>
<td>15% $\downarrow$ 5%^{+\text{stat}}</td>
</tr>
<tr>
<td>Finite volume corrections</td>
<td>6.0% $\downarrow$ 2%</td>
<td>6.5% $\downarrow$ 2%</td>
</tr>
<tr>
<td>Partial quenching</td>
<td>3.5% $\downarrow$ 0%</td>
<td>1.7% $\downarrow$ 0%</td>
</tr>
<tr>
<td>Renormalization</td>
<td>1.8% (?)</td>
<td>5.6% (?)</td>
</tr>
<tr>
<td>Unphysical kinematics</td>
<td>0.4% =</td>
<td>0.8% =</td>
</tr>
<tr>
<td>Derivative of phase shift</td>
<td>0.97% =</td>
<td>0.97% =</td>
</tr>
<tr>
<td>Wilson coefficients</td>
<td>6.6% =</td>
<td>6.6% =</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>18% $\downarrow$</td>
<td>19% $\downarrow$</td>
</tr>
</tbody>
</table>
Cancellation of contractions in \( \text{Re}(A_2) \)

The dominant contribution comes from \((27,1)\) operator. \((27,1)\) operator is proportional to the sum of the following two contractions:
Cancellation of contractions in $\text{Re}(A_2)$
Cancellation of contributions to $\text{Re}(A_2)$ appears in all our simulations.

Investigation of $\text{Re}(A_0)$ at threshold (arXiv:1212.1474) shows that all contributions have the same sign resulting in small enhancement of $\text{Re}(A_0)$.

The main mechanism behind $\Delta I = 1/2$ rule seems to be a cancellation in $\text{Re}(A_2)$!
Conclusions

- Many lattice parameters need to be fine tuned
- (Preliminary) Results from $48^3$ and $64^3$ ensembles are consistent with $32^3$ DSDR results
- Systematic errors due to lattice artefacts are smaller than anticipated
- Cancellation in $\text{Re}(A_2)$ has been confirmed in both ensembles
- Systematic errors need to be estimated more carefully
Thank you for your attention!