*Radial Quantization for Conformal Field Theories on the Lattice**

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Boston University
Lattice 2013 Mainz, August 1, 2013

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

* Collaborators: Michael Cheng, George Fleming and Herbert Nauberger

Motivation

- (near) Conformal Field Theories are important
  - BSM walking technicolor
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

- Lattice difficulty: scales are exponentially divergent.

- Linear Hypercubic vs Exponential Radial Lattice

$$a < \Delta r < L \quad \text{vs} \quad a < \Delta log(r) < L$$

Early History

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. This leads to an operator formulation for Euclidean quantum field theory. A covariant thermodynamics is developed, with which the Hagedorn spectrum can be obtained, given further hypotheses. The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For $d > 2$ these correspond however, to curved spaces. The result is verified for the spherical model

Narrative

I. Our first attempt

-- *Lattice Radial Quantization: 3D Ising* R.C.B., G.T. Fleming and H. Neuberger, , Phys. Lett. B 721 (2013) 299

II. What worked and what failed

III. Finite Elements to the rescue ?

IV. Future hopes and dreams

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Lattice radial quantization: 3D Ising

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ABSTRACT

Lattice radial quantization is introduced as a nonperturbative method intended to numerically solve Euclidean conformal field theories that can be realized as fixed points of known Lagrangians. As an example, we employ a lattice shaped as a cylinder with a 2D Icosahedral cross-section to discretize dilatations in the 3D Ising model. Using the integer spacing of the anomalous dimensions of the first two descendants ($l = 1, 2$), we obtain an estimate for $\eta = 0.034(10)$. We also observed small deviations from integer spacing for the 3rd descendant, which suggests that a further improvement of our radial lattice action will be required to guarantee conformal symmetry at the Wilson–Fisher fixed point in the continuum limit.

Radial Quantization

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop
Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$''time'' \quad \tau = log(r) \; , \; ''mass'' \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

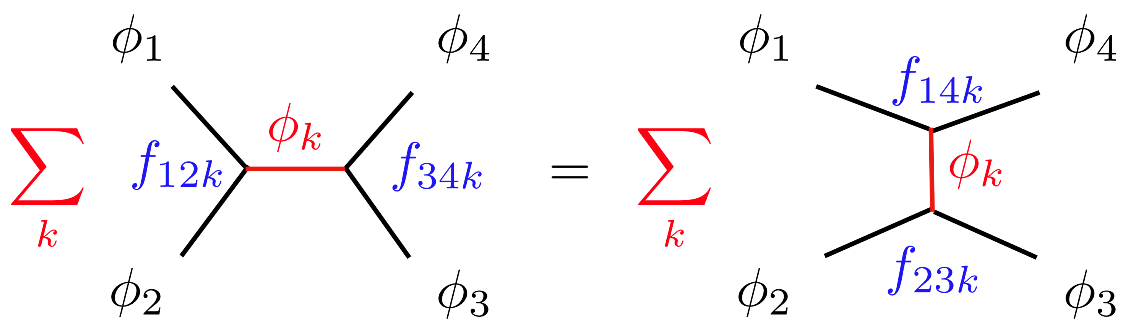
CFT are highly constrained

More than hyper scaling
(scale invariance).
2 and 3 point correlators
are determined.

$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}}\mathcal{O}_k(0)$$

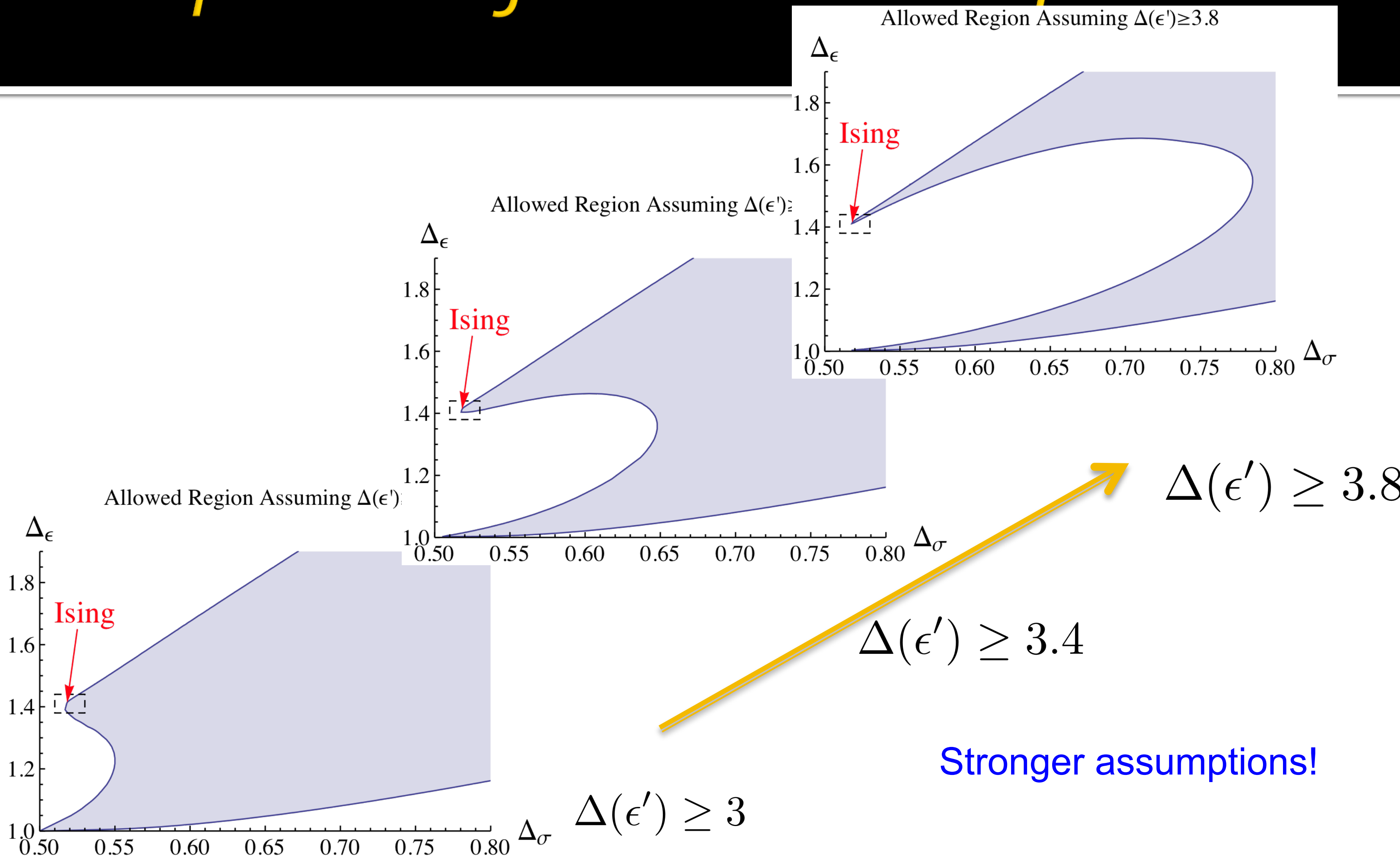
OPE & factorization completely fixed the theory*
(i.e. Data: spectral + couplings to conformal blocks)



complete sum over
the conformal blocks
"partial waves".
Only "tree" diagrams!

* "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

*Inequalities from Bootstrap**



•"Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

$$\begin{aligned}
r_1^{\Delta} r_2^{\Delta}\langle \phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^{\Delta}} \\
&\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\
&= Ce^{-\tau\Delta}
\end{aligned}$$

With $|x_1 - x_2|^2 = r_1 r_2[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

Lesson #1:
Radial Quantization is not necessarily Conformal

- Consider large N 2d O(N) sigma model:

  (what happens to conformal anomaly?)

  -- Quantize on R^2: Get Lorentz invariant theory

  -- Quantize on $R\times S^1$: Get Dilation invariant

  But NOT both!

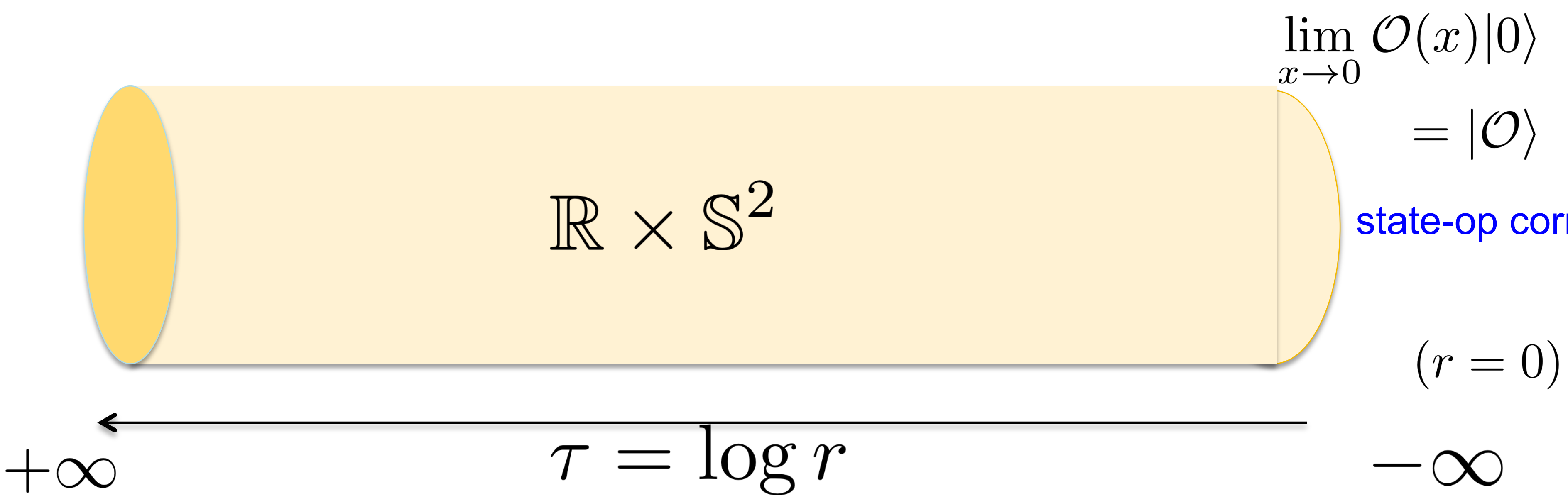
What happens to Radial Quantization? Try it?

$$\langle \Phi^{*}{}^{i}_{l'}(\tau)\Phi^{j}_{l}(\tau')\rangle = \delta^{ij}\delta_{ll'}\frac{e^{-\sqrt{l^2+\mu^2}|\tau-\tau'|}}{2\sqrt{l^2+\mu^2}} \ .$$

LESSON: Descendants don't have integer-spaced descendants. Consequently, we cannot construct translation generators satisfying the correct commutation relations with dilatations in the sector generated by the action of $\vec{\Phi}$ on the vacuum. The deviation of the dilatation spectrum from equal spacing is small if $l \gg \mu$. Because inversion has also been preserved in the quantization, if translations could be realized, special conformal transformations would come in automatically and the full conformal group would be realized. Because of rotation invariance, only one linear combination of translations needs to be considered in detail.

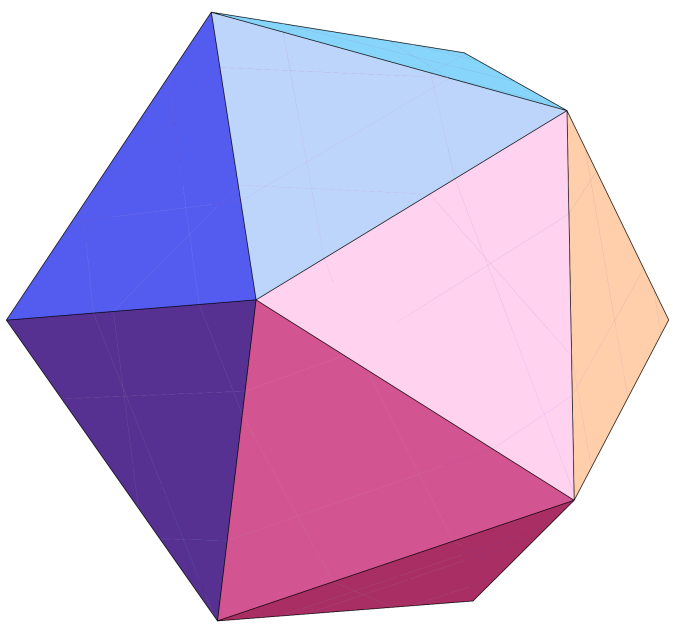
Lesson #2: 3-d Ising at Wilson-Fisher FP

$$Z_{Ising} = \sum_{\sigma(x,t)=\pm 1} e^{\beta \sum_{t,\langle x,y \rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t+1,x)\sigma(t,x)}$$

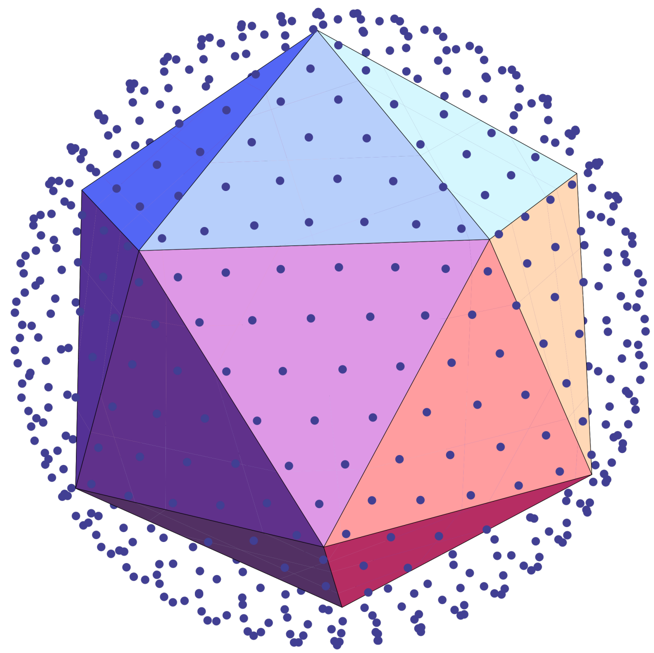


Order s Refined Triangulated Icosahedron

$s = 1$

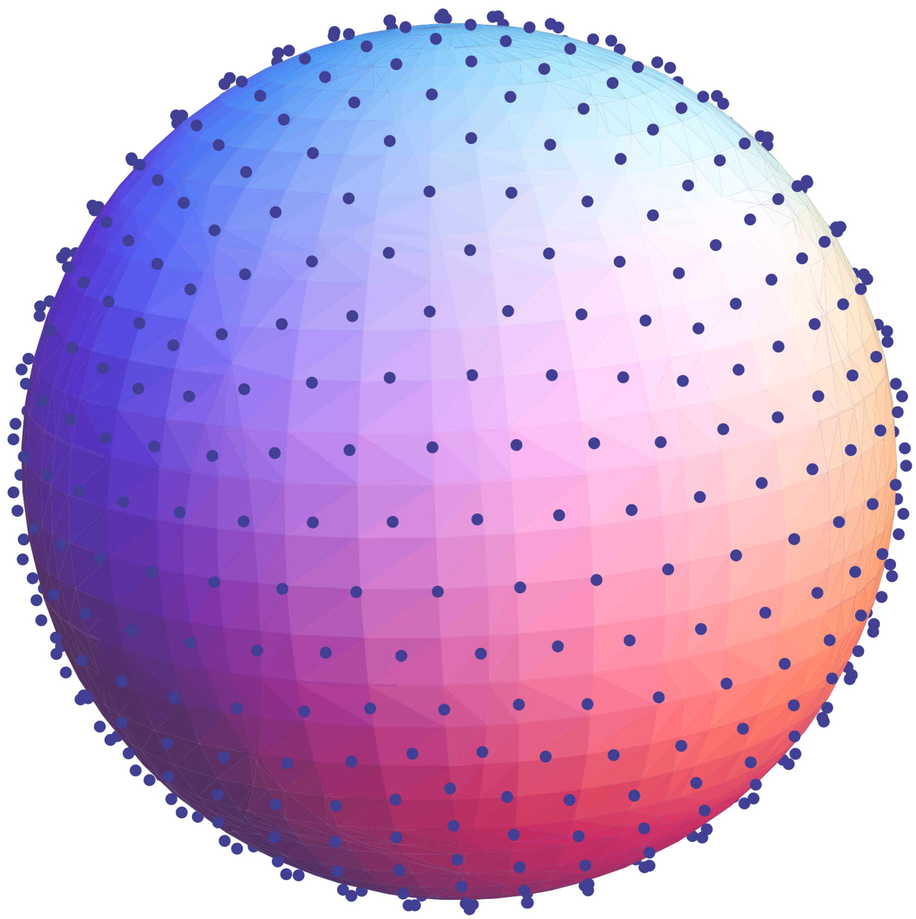


$s = 8$



l = 0 (A),1 (T1) , 2 (H) are irreducible 120
Iscosahedral subgroup of O(3)

Fixed t lattice are s refined Icosahedrons



s = 8

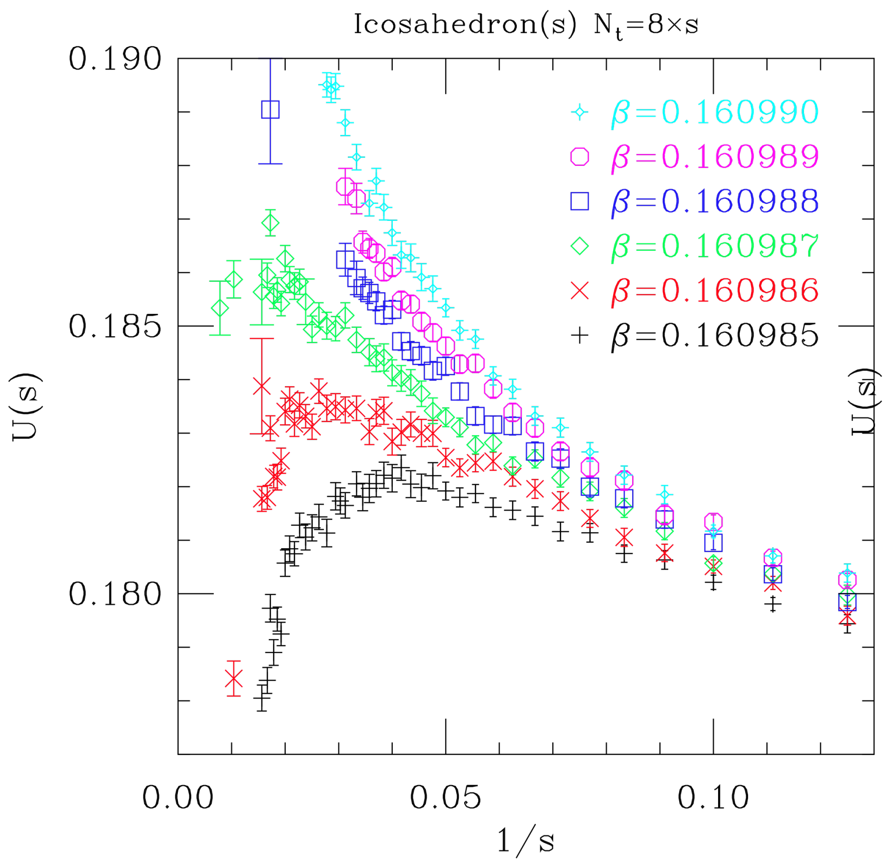
vertices:
N = 10 + 2*s*s = 138

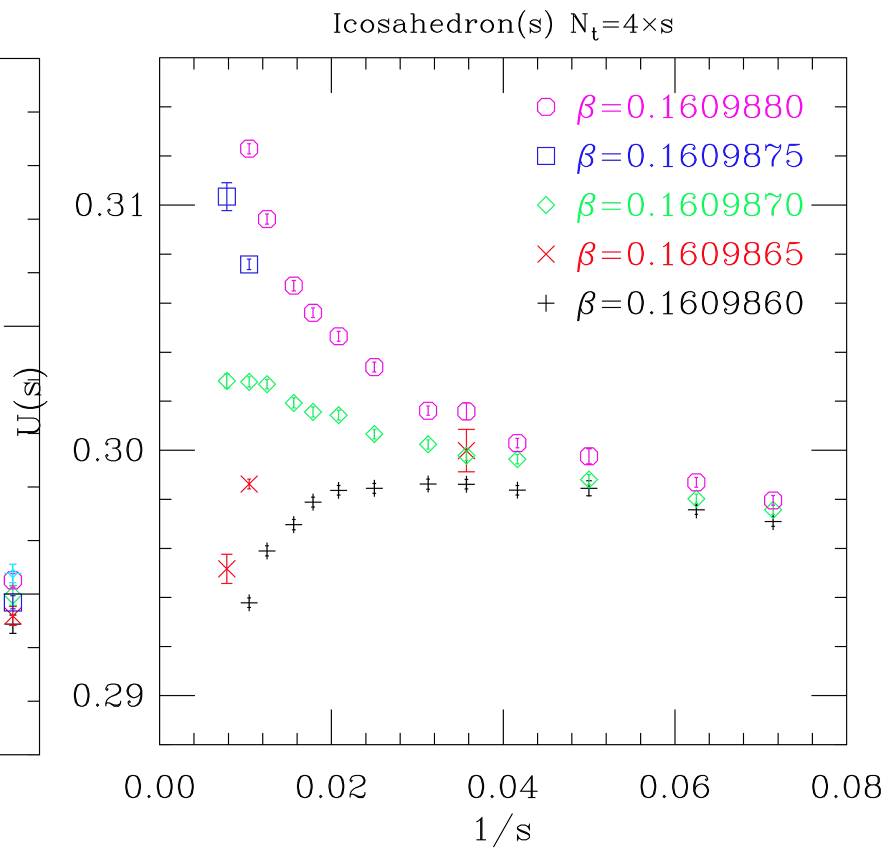
edges:
E = 3*N – 6

faces:
F = E – N + 2 = 2*N - 4

$$\text{Continuum limit is } s \to \infty \text{ at } \beta = \beta_{critical}$$

Determining beta_critical





$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

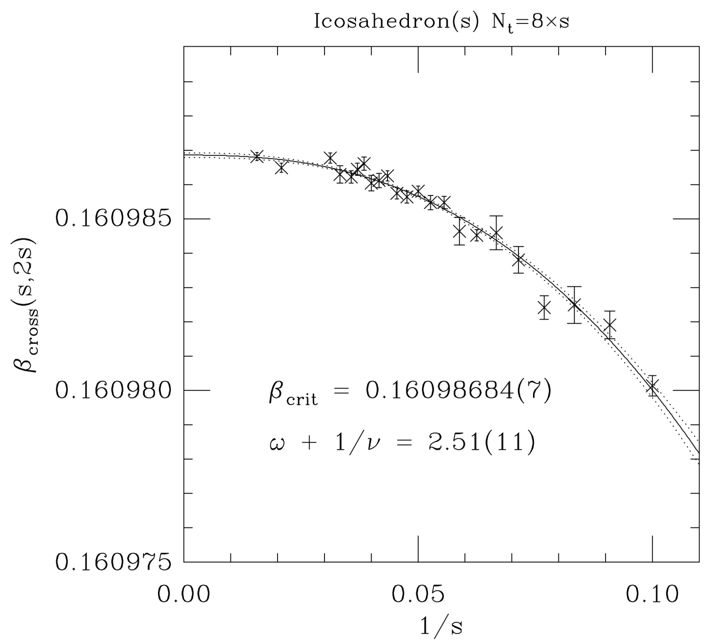
$$\beta_{crit} = 0.16098703(3)$$

Determining beta_critical

$$
\begin{aligned}
U_L(\beta) &= \widetilde{U}(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots \\
&= U^* + a_1(\beta-\beta_c)L^{1/\nu} + b_1 L^{-\omega} + b_2 L^{-\gamma/\nu} + \cdots
\end{aligned}
$$

$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

$$\beta_{crit} = 0.16098703(3)$$



Improved Operators

$$\widetilde{\sigma}_{lm}(t) = \sum_x \sqrt{\omega_x}\, Y_{lm}(\theta_x, \phi_x)\sigma(t, x)$$

Area of spherical triangle
projected on the units sphere

$$\lim_{s\to\infty} \sum_{x=1}^{2+10s^2} \omega_x\, Y^*_{l'm'}(\theta_x, \phi_x) Y_{lm}(\theta_x, \phi_x) = 4\pi\delta_{l'l}\delta_{m'm}$$

Orthonormality
to 10^{-3}, $4 * 10^{-5}$, $5 * 10^{-6}$
at $s = 16, 64, 256$ respectively

Observables

$$C_{lm}(t) = \sum_{t_0,x,y} \sqrt{\omega_x \omega_y}\, Y^*(\hat{x}) \,\langle \sigma(t+t_0,x)\sigma(t_0,y)\rangle\, Y_{lm}(\hat{y})$$

cosh fit:

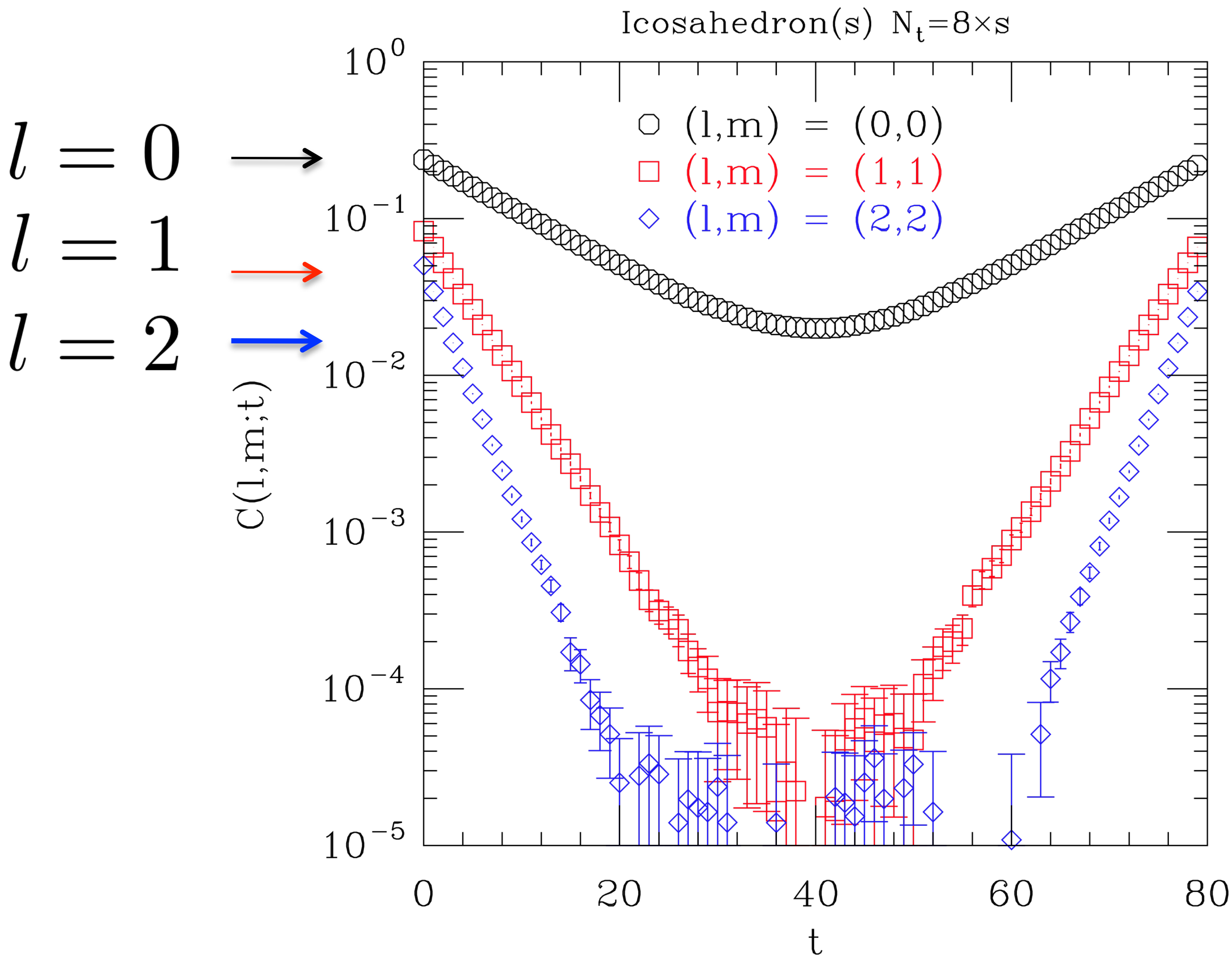
$$C_{lm}(t) = C[e^{-m_l t} + e^{-m_l(N_t-1-t)}]$$

$$\text{for} \quad t = 0, \cdots, N_t - 1\,,\; l = 0,1,2\,,\; m = -l, \cdots, l$$

$$m_l = \frac{c}{s}\Delta_l \qquad\qquad \Delta_l = \frac{1}{2} + \frac{\eta}{2} + l$$

After you adjust c = $\texttt{speed of light}$ so $\Delta_{l+1} - \Delta_l = 1$

Early result for C(t)



Fitting correlators

- Discrete states have exact cosh correlators

$$C_l(t) = A_l \cosh(-\mu_l(t - T/2))$$

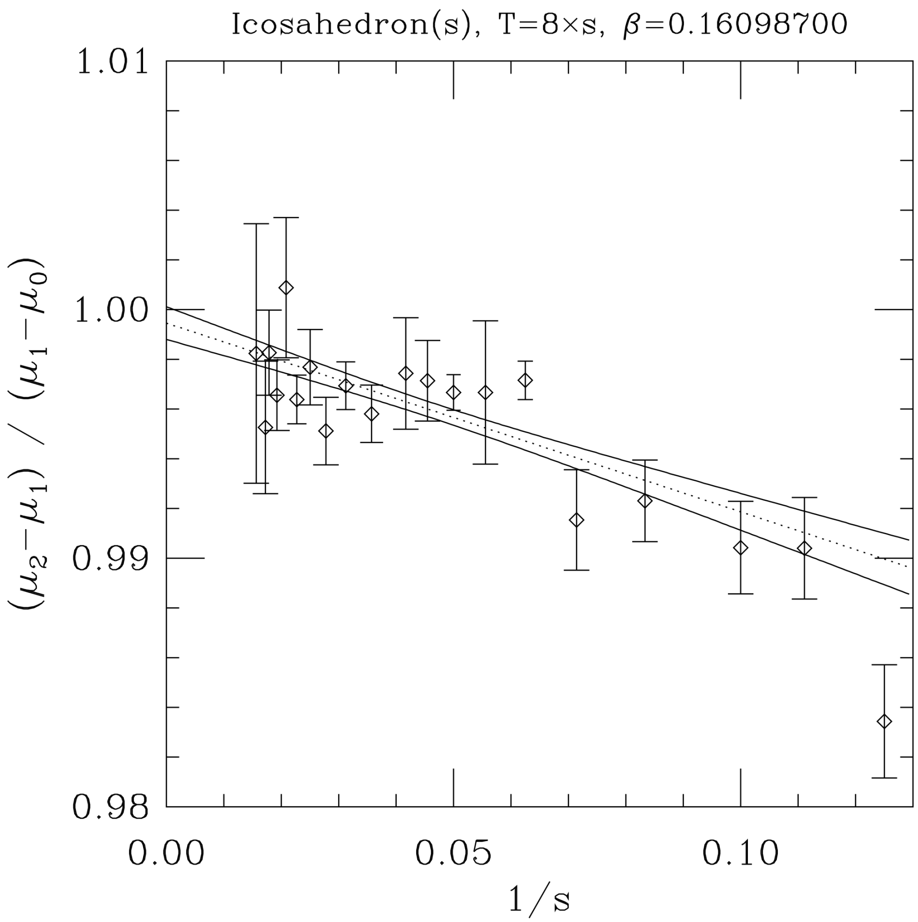
- Transform to k–space

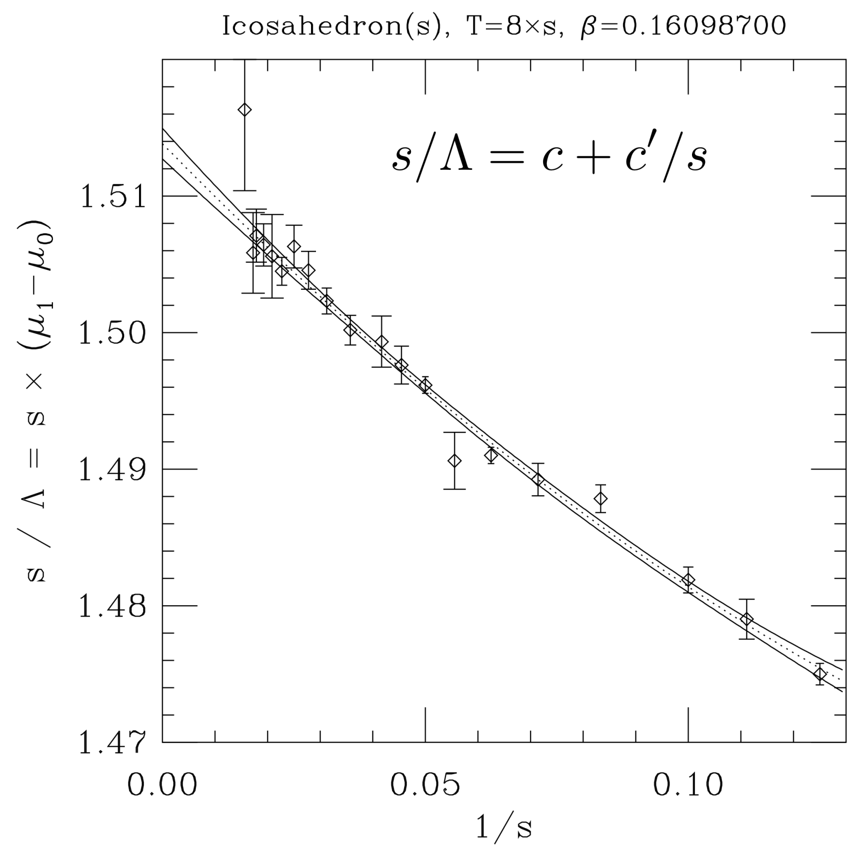
Disconnected piece

$$\begin{aligned}\widetilde{C}_l(k) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{itk}\, C_l(t) \\ &= c_0\delta_{l,0}\delta_{k,0} + a_l \frac{(1-e^{-\mu_l T})\sinh(\mu_l)}{\sinh^2(\mu_l/2)+\sin^2(k/2)}.\end{aligned}$$

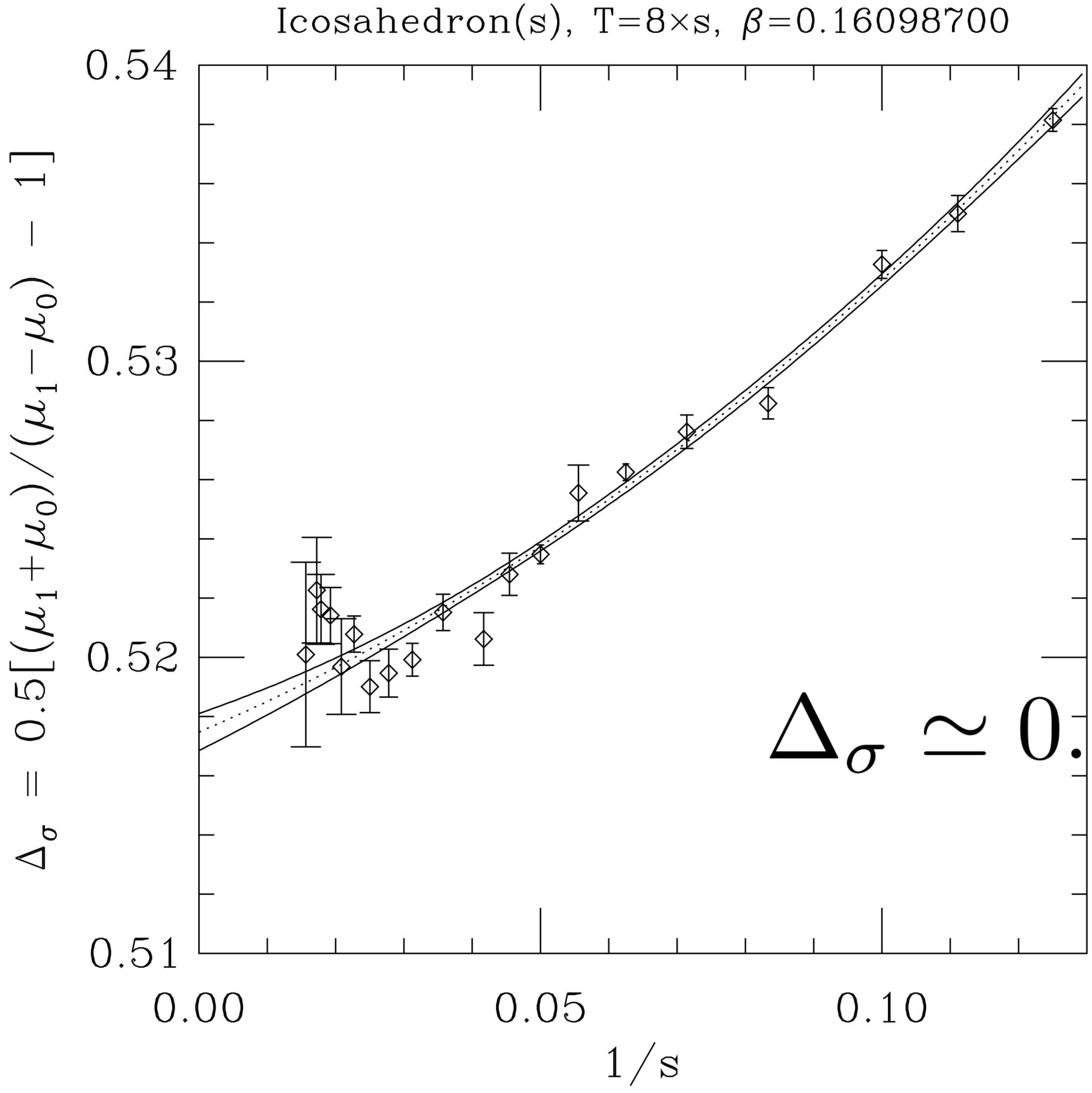
- Good fits required 3 mass

Check Descendant Relation & rescale "log(r)"





Current Fit: $\Delta_\sigma = 0.5175(6)$



$$\Delta_\sigma \simeq 0.5175$$

Primary operators 3-d Ising Model

Operator	Spin l	$\mathbb{Z}$	Δ	Exponent
s	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
s'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	$\Delta = 3$
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\mathrm{NR}}$

Low-lying primary operators of the 3D Ising model at criticality.

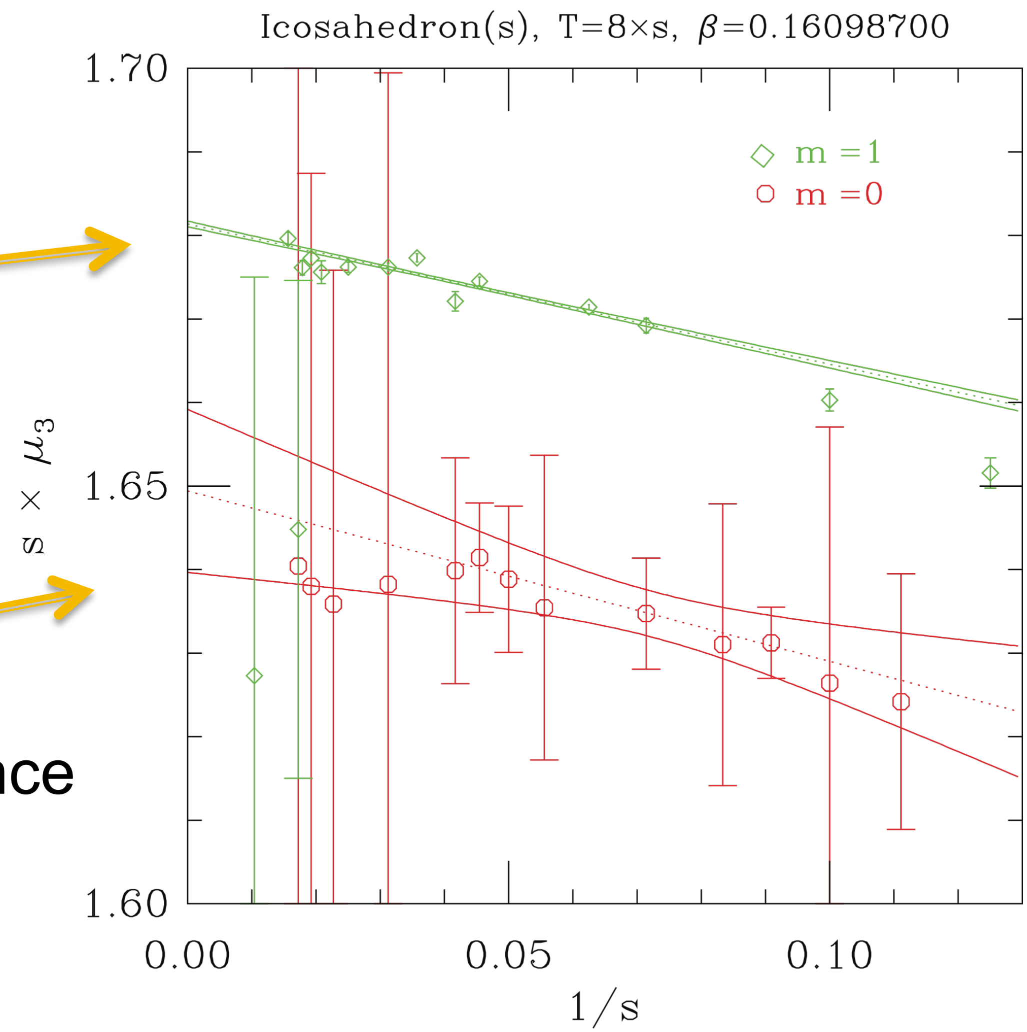
Primary l = 0 $$[K_\mu, \mathcal{O}(0)] = 0$$

Descendants l > 0 $$\mathcal{O}_{l+1}(x) = [P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}_l(x)$$

Failure to recover O(4,1) of l = 3?

G rep

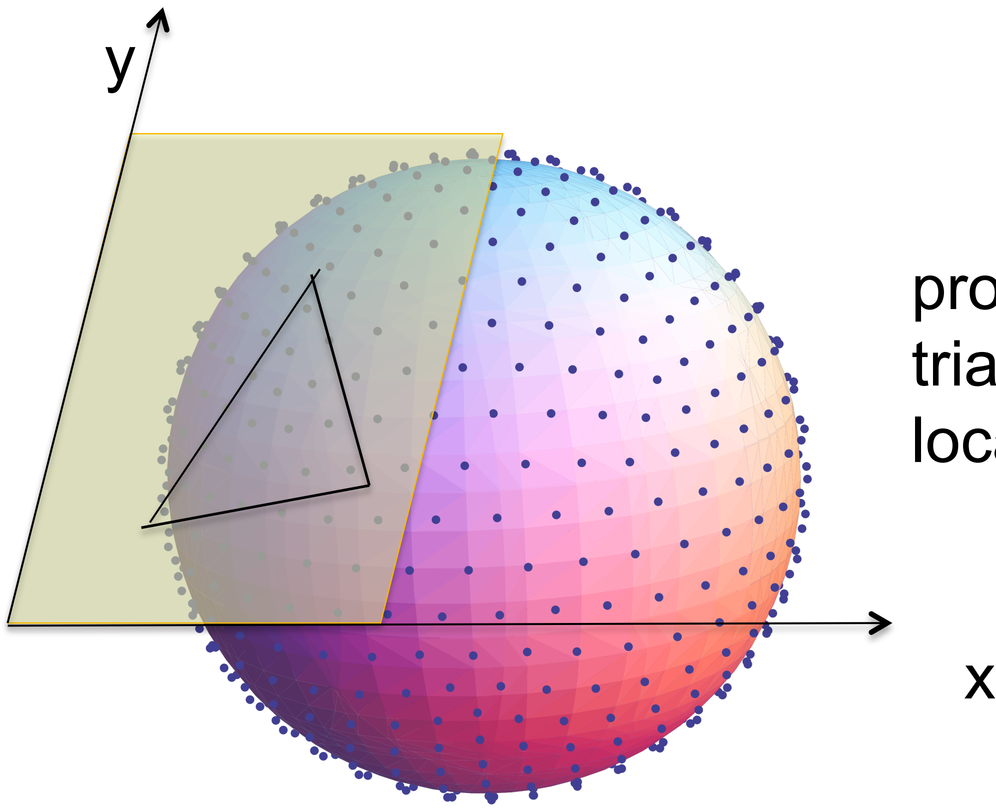
T2 rep



Apparent lack of convergence to a single O(3) irreducible representation for l = 3

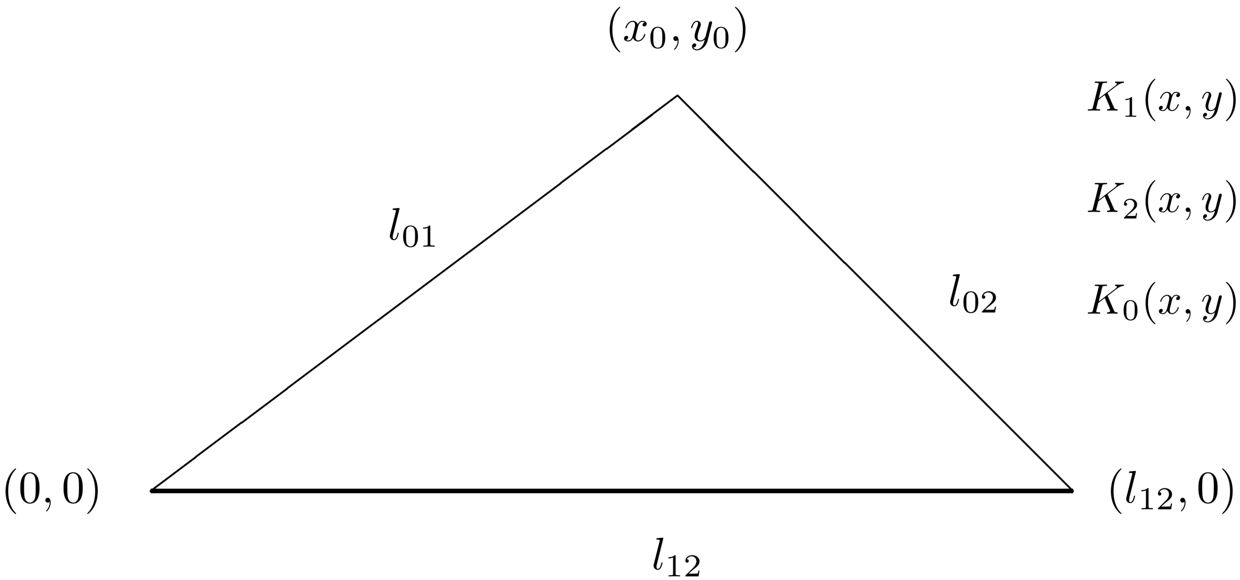
phi 4th and Finite Element Method

$$Z = \int \mathcal{D}e^{-\beta K_{ij}(\phi_i - \phi_j)^2 - \lambda\omega_i(\phi_i^2 - 1)^2}$$



project spherical
triangle onto
local tangent plane

Linear Finite Element Method for triangulate Manifold



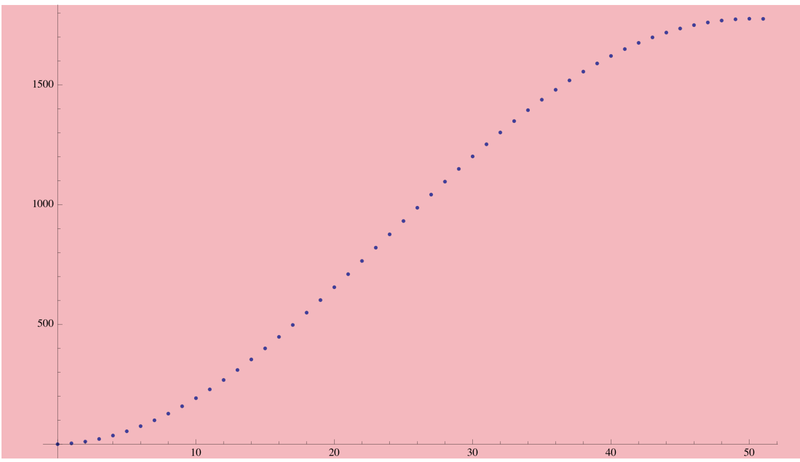
$$
\begin{aligned}
K_1(x,y) &= [l_{12} - x - \frac{(l_{12} - x_0)y}{y_0}]/l_{12} \\
K_2(x,y) &= [x - \frac{x_0 y}{y_0}]/l_{12} \\
K_0(x,y) &= \frac{y}{y_0}
\end{aligned}
$$

triangle on the tangent plane

On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

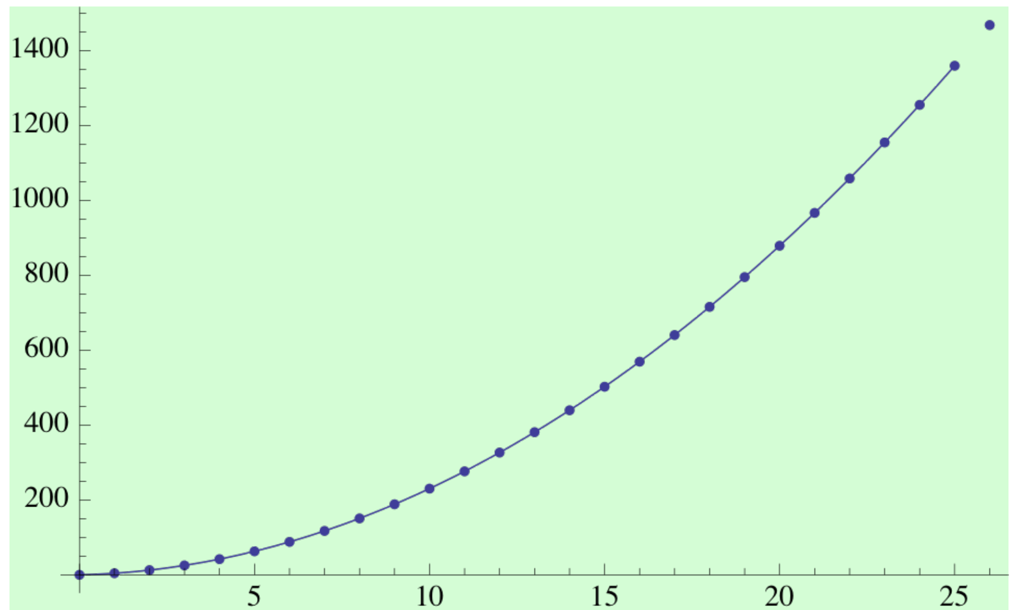
$$
\int_{A_{012}} dx dy \partial_\mu \phi(x,y) \partial_\mu \phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]
$$

Spectrum of FEM Laplacian on a sphere



s =8

s = 512



$$2.09439l + 2.09439l^2 - 5.75 * 10^{-6}l^3 - 2.95833 * 10^{-6}l^4$$

Comment on FEM Radial CFT

(1) There are theorems such errors are $O(a^{n+1})$
"for piecewise FEM of order n with max diameter a for all simplices with bounded angles".

(2) The local derivative expansion does not give

$$\nabla^2\phi + O(a^2)$$

BUT the spectra or the operator converges if
well separated from the 1/a cut-off (like Wilson Flow?)

(3) Perturbative (epsilon expansion) proof is not impossible?
Non-perturbative F.P. is very difficult except numerically?

The simulation program is written and being tested

(1) Monte Carlo is a “standard” mixture of metropolis, over relax and Wolff methods from:

(i) M. Hasenbusch,``A Monte Carlo study of leading order scaling corrections of phi**4 theory on a three-dimensional lattice” J.Phys. A 32 (1999) 4851 *

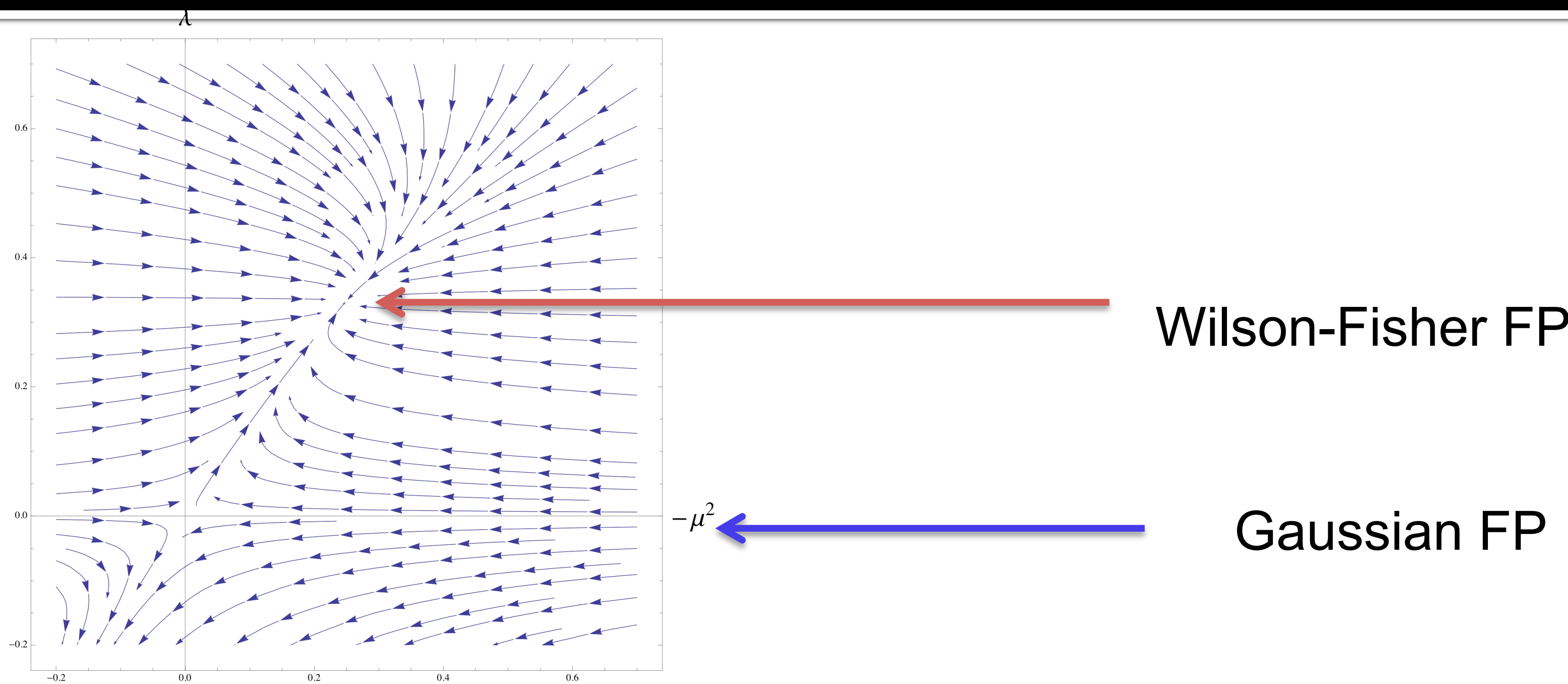
(ii) Ulli Wolff, “Collective Monte Carlo Updating for Spin Systems PRL 62: 361 (1989)

(iii) R.C.B. and P. Tamayo, “Embedded Dynamics for phi4 Theory”, PRL 62:1087(1989)

(2) Will compute higher primaries, even Z2 sector, Energy momentum tensor, 2-2 partial wave amplitude etc.

(3) The code can run any graph, so we will replace sphere by torus to reproduce phi 4^{th} numbers from Hasenbusch et al

GR flow in Epsilon Expansion



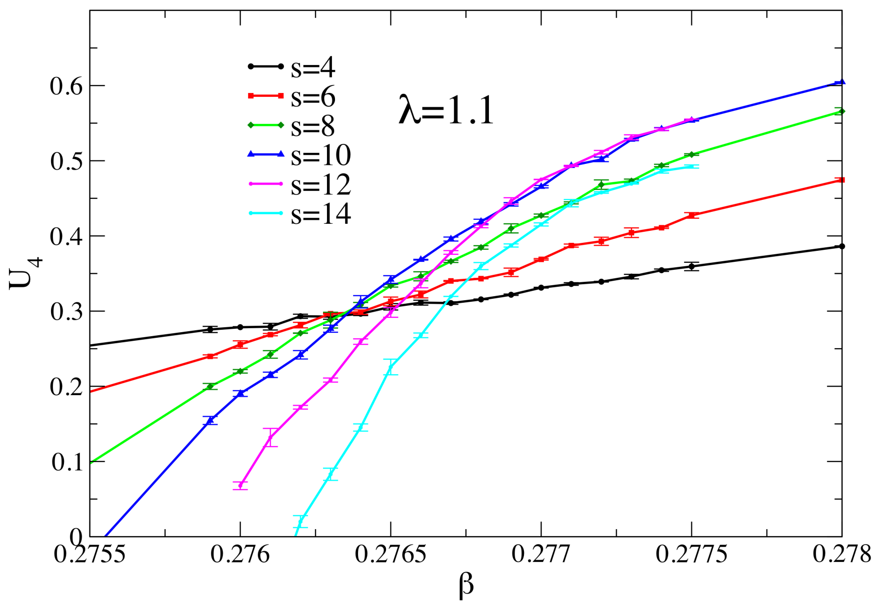
$$\beta_g = \epsilon g - \frac{3}{16\pi^2} g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g;)$$

$$\beta_{\mu^2} = 2\mu^2 + ag + \frac{9}{16\pi^2} g\mu^2 + O(\mu^4) \qquad \lambda = 4g/4!$$

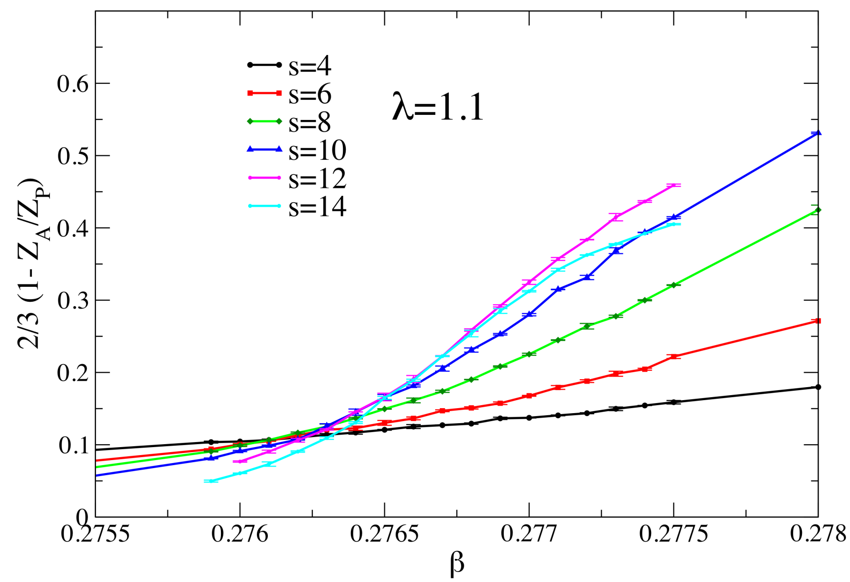
Just starting to get Numerical Results!

Still verifying code but ... here are two PREMINARY tests~

Binder Cummulant



Antiperiodic/Periodic



Future Challenges & Directions

- Many extensions are interesting once this is "proven" to work for simple models:
  - Easy problems:
    - O(N) model in 3-d compared with large N
    - Strengthen bootstrap inequalities ?
  - Hard Problems:
    - Gauge fields (with discrete Chirstoffel connection)?
    - Fermions (with discrete spin connection) ?
    - Flow from UV to conformal IR fixed points for BSM? (Dilation operator is only asymptotically const in "time".)

Extra Slides

Euclidean Conformal Field Theories

O(d+1,1) adds Dilations and Inversion to Poincare transformations

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

$$K_\mu : (inv \to trans \to inv)$$

Algebra:

$$[K_\mu, \mathcal{O}(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

Improved cluster Estimator

Swendsen-Wang: Real space

$$g(x-y) = \langle s_x s_y \rangle \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \sum_{C_i} \Delta_{C_i}(x)\Delta_{C_i}(y)$$

$$\Delta_C(x) = \quad 1 \text{ if } x \in C \text{ else } 0$$

Wolff single cluster

$$\widetilde{g}_{lm}(k) \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \frac{1}{|C|} \Big| \sum_{t,x\in C} e^{i2\pi kt/L_t} Y_{lm}(\Omega_x) \Big|^2$$

Note: All to All O(V) improved estimator in Momentum space *

*C. Ruge, P. Zhu and F. Wagner Physica A (1994) 431:

Numerical Test (so far)

- Equal spacing test of descendants:

$$\frac{\mu_2 - \mu_1}{\mu_1 - \mu_0} = 0.999(1)$$

- "Speed of light" $c = 1.5105(7)$

- But critical point $\beta_{crit} = 0.16098703(3)$

- Current anomalous dimensions (more soon)
  - from Binder: $\omega + 1/\nu = 2.51(11)$
  - from corr: $\Delta_\sigma = 1/2 + \eta/2 = 0.5175(6)$
  - Simulation are on going to reduce errors