# The Integrable Bootstrap Program at Large N and its Applications in Gauge Theory

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# The Principal Chiral Sigma Model (PCSM)

$$\text{Action}: S = \frac{N}{2g}\int d^2x \, \mathrm{Tr}\partial_\mu U^\dagger(x)\partial^\mu U(x),$$

$$U(x) \in SU(N):$$

$$SU(N)\times SU(N)\,\text{symmetry}: \quad U(x) \to V_L U(x) V_R, \; V_{L,R} \in SU(N).$$

Associated Noether currents:

$$j^L_\mu(x)^c_a = \frac{-iN}{2g^2}\partial_\mu U_{ab}(x)\, U^{*\,bc}(x),$$

$$j^R_\mu(x)^d_b = \frac{-iN}{2g^2} U^{*\,da}(x)\partial_\mu U_{ab}(x)$$

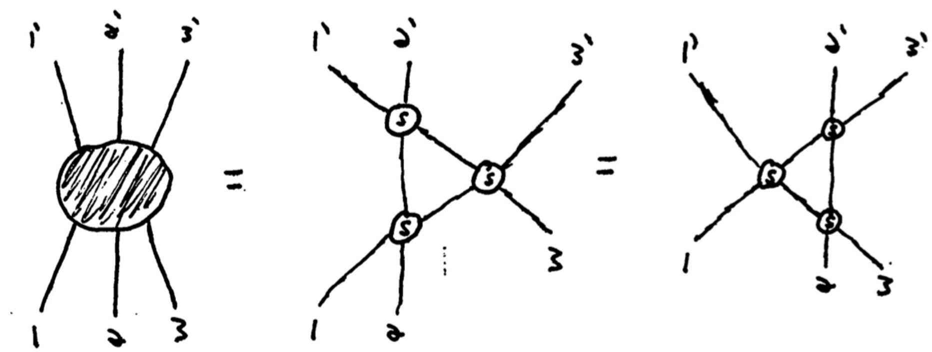
Theory of asymptotically free massive particles, with left and right color.

We work in the 'tHooft (planar) limit.

# Integrable Quantum Field Theory

Integrability: Equal number of conservation laws and degrees of freedom (infinite in QFT)

In Quantum field Theory there is no particle production. Set of momenta is conserved $\{p\}_{\rm in} = \{p\}_{\rm out}$. Scattering is factorizable.



Yang-Baxter equation

General form factor of the operator $\mathcal{O}(x)$:

$$\langle 0|\mathcal{O}(x)|\text{state with particles and antiparticles}\rangle$$

# The S-Matrix

Particles and antiparticles have two color charges (color dipoles). Two-particle S-matrix determined by Yang-Baxter equation, unitarity and crossing symmetry.

$$_{\rm out}\langle P,\theta_1',c_1,d_1;P,\theta_2',c_2,d_2|P,\theta_1,a_1,b_1;P,\theta_2,a_2,b_2\rangle_{\rm in}$$

$$= S(\theta,N)\left(\delta^{c_1}_{a_1}\delta^{c_2}_{a_2}-\frac{2\pi i}{N\theta}\delta^{c_2}_{a_1}\delta^{c_1}_{a_2}\right)\times\left(\delta^{d_1}_{b_1}\delta^{d_2}_{b_2}-\frac{2\pi i}{N\theta}\delta^{d_2}_{b_1}\delta^{d_2}_{b_2}\right)\langle\theta_1'|\theta_1\rangle\langle\theta_2'|\theta_2\rangle$$

$$\theta = \text{rapidity} : E = m\cosh\theta,\;\; p = m\sinh\theta,\;\; E^2 = p^2+m^2$$

$$\text{rapidity difference}\,\theta=\theta_{12}=\theta_1-\theta_2$$

$$\text{At large}\,N:\;\; S(\theta,N)=1+\mathcal{O}\left(\frac{1}{N^2}\right).$$

Particle-antiparticle related by crossing $\theta\to\hat{\theta}=\pi i-\theta$.

# The current operator ansatz

$$\langle 0|j^L_\mu(x)_{a_0 a_{2M+1}}|A_1;\ldots;A_M;P_{M+1};\ldots;P_{2M}\rangle$$

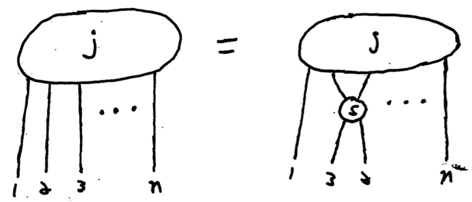
$$= [p_1+\cdots+p_M-(p_{M+1}+\cdots+p_{2M})]_\mu \frac{e^{-ix\cdot\sum p}}{N^{M-1}} \sum_{\sigma,\tau\in S_M} F_{\sigma\tau}(\theta_1,\ldots,\theta_{2M})$$

$$\times \Big[\prod_{j=0}^M \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^M \delta_{b_k b_{\tau(k)+M}} \\ -\frac{1}{N}\delta_{a_0 a_{2M+1}} \delta_{a_{l_\sigma} a_{\sigma(0)+M}} \prod_{j=1,\, j\neq l_\sigma} \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^M \delta_{b_k b_{\tau(k)+M}}\Big]\,,$$

$\sigma \in S_M$, takes $\{1,2,\ldots,M\}$ to $\{\sigma(1),\sigma(2),\ldots,\sigma(M)\}$

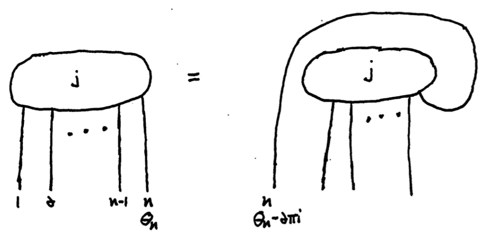
# Smirnov's form factor axioms

Scattering Axiom (Watson′s theorem)




$$\langle 0|j|P_2, A_1\rangle = S^{12}_{AP}\langle 0|j|A_1, P_2\rangle$$

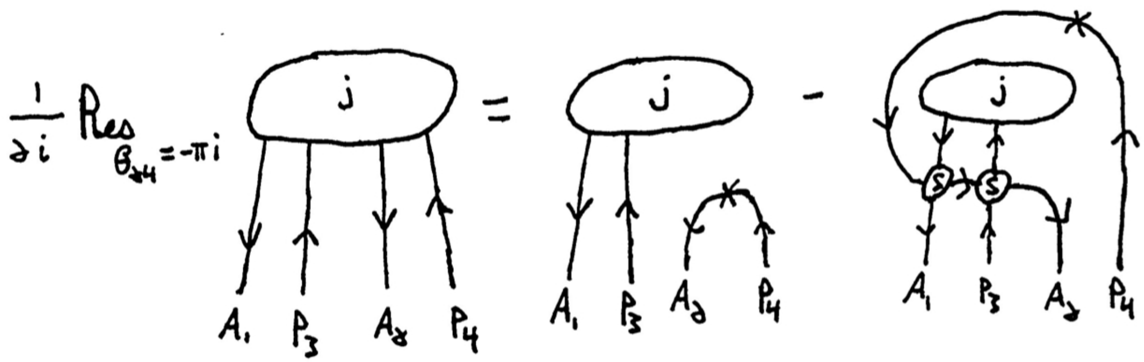
Periodicity axiom




$$\langle 0|j|A_1(\theta_1), P_2(\theta_2)\rangle = \langle 0|j|P_2(\theta_2 - 2\pi i), A_1(\theta_1)\rangle$$

# Smirnov's form factor axioms

## Annihilation pole axiom




The antiparticle $A_2$ and the particle $P_4$ can annihilate. The four particle form factor needs to have an annihilation pole at $\theta_{24} = -\pi i$.

## Underlying Abelian Structure at Large N

The excitations in the incoming state of the form factor only interact with each other if they have color indices contracted together.

We can order incoming particle such that they only interact with their two nearest neighbors. Particles now have the simple commutation relation

$$\mathfrak{A}^\dagger(\theta_j)\mathfrak{A}^\dagger(\theta_k) = \frac{\theta_k - \theta_j + \pi i}{\theta_k - \theta_j - \pi i}\mathfrak{A}^\dagger(\theta_k)\mathfrak{A}^\dagger(\theta_j), \text{ if } k = j+1$$

Behaves like colorless Abelian particles at large $N$.

This is not related to integrability, but to the large $N$ limit.

Is a nonintegrable large $N$ bootstrap possible?

# Solution from Smirnov's axioms

$$F_{\sigma\tau}(\{\theta\}) = \frac{g_{\sigma\tau}}{\prod_{j=1, j\neq l_\sigma}^{M}\left(\theta_j - \theta_{\sigma(j)+M} + \pi i\right)\prod_{k=1}^{M}\left(\theta_k - \theta_{\tau(k)+M} + \pi i\right)},$$

From the annihilation pole axiom:

$$g_{\sigma\tau} = \begin{cases} 2\pi i (4\pi)^{M-1}, & \text{for } \sigma(j) \neq \tau(j), \text{ for all } j \\ 0 \quad , & \text{else} \end{cases}$$

unphysical double poles go away!

## The two-point function

We can calculate exactly the current-current correlator,

$$W_{\mu\nu}(x)_{a_0 c_0 e_0 f_0} = \frac{1}{N}\langle 0| j^L_\mu(x)_{a_0 c_0}\, j^L_\nu(0)_{e_0 f_0}|0\rangle$$

$$= \frac{1}{N}\sum_{\Psi}\langle 0| j^L_\mu(x)_{a_0 c_0}|\Psi\rangle\,\langle \Psi| j^L_\nu(0)_{e_0 f_0}|0\rangle$$

$\langle 0| j^L_\mu(x)_{a_0 c_0}|\Psi\rangle$ are the form factors we know

$$W_{\mu\nu}(x)_{a_0 c_0 e_0 f_0} = \sum_{M=1}^{\infty}\int\left(\prod_{j=1}^{2M}\frac{d\theta_j}{4\pi}\right) e^{-ix\sum p} 4\pi^2 (4\pi)^{2M-2}$$

$$\times[p_1+p_3+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\mu [p_1+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\nu$$

$$\times(\delta_{a_0 e_0}\delta_{c_0 f_0} - \frac{1}{N}\delta_{a_0 c_0}\delta_{e_0 f_0})\prod_{j=1}^{2M-1}\left[\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}\right]$$

# The energy-momentum two-point function

$$W^T_{\mu\nu\alpha\beta}(x) = \frac{1}{N^2}\langle 0|T_{\mu\nu}(x)T_{\alpha\beta}(0)|0\rangle$$

$$= \sum_{M=1}^{\infty} \frac{\pi}{8} \int \left( \prod_{j=1}^{2M} d\theta_j \right) e^{-ix\sum p}$$

$$\times[p_1+p_3+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\mu[p_1+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\nu$$

$$\times[p_1+p_3+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\alpha[p_1+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\beta$$

$$\times\frac{1}{[(\theta_1-\theta_{2M})^2+\pi^2]}\prod_{j=1}^{2M-1}\frac{1}{[(\theta_j-\theta_{j+1})^2+\pi^2]}$$

# Anisotropic QCD

Longitudinal Rescaling: $x^{0,1} \to \lambda x^{0,1},\ x^{2,3} \to x^{2,3}$

$$A_{0,1} \to \lambda^{-1} A_{0,1},\ A_{2,3} \to A_{2,3}$$

$$H = H_0 + \lambda^2 H_1 + \lambda^2 H_2$$

$$= \left[\int d^3x \left(\frac{g^2}{2} E_\perp^2 + \frac{1}{2g^2} B_\perp^2\right)\right] + \lambda^2 \left[\int d^3x \frac{g^2}{2} E_1^2\right] + \lambda^2 \left[\int d^3x \frac{1}{2g^2} B_1^2\right]$$

Examine the $\lambda \to 0$ limit

no $H_2$ in 2+1 dimensions

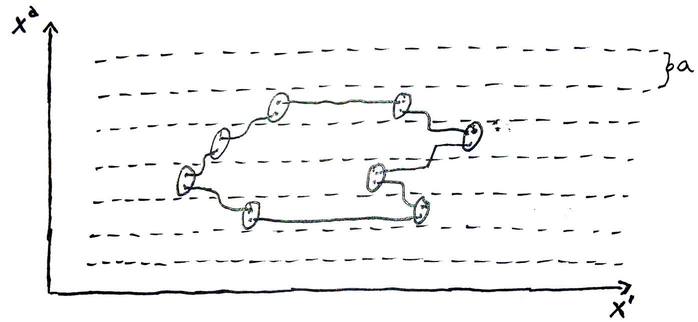
# Anisotropic Lattice, 2+1 dimensions

Gauge choice: $A_0 = A_1 = 0$, make $x^2$ direction discrete.

$$H_0 = \sum_{x^2} H_{PCSM}(x^2), \text{ with } SU(N) \text{ field } U(x) = e^{iaA_2(x)}$$

$$H_1 = -\sum_{x^2} \int dx^1 \int dy^1 \frac{\lambda^2}{4g_0^2 a^2} |x^1 - y^1|$$

$$\times [j_0^L(x^1, x^2) - j_0^R(x^1, x^2 - a)] \times [j_0^L(y^1, x^2) - j_0^R(y^1, x^2 - a)]$$




We compute corrections from $\langle \Psi' | H_1 | \Psi \rangle$ with our form factors

# Form factor perturbation theory

We can define a "transfer matrix" to evolve the system in the $x^2$ direction:

$$T_{x^2,x^2+a} = e^{-\frac{1}{2}H_0(x^2)-\frac{1}{2}H_0(x^2+a)-H_1(x^2,x^2+a)}$$

Truncated spectrum approach: Organize states of $H_0$ by energy $|1\rangle$, $|2\rangle$, $|3\rangle, \ldots,$ $|n\rangle$. Discretize by putting in a box.

$E_n$ is the truncation energy.

The (now finite) matrix $T_{jk} = \langle j|T_{x^2,x^2+a}|k\rangle$ can be diagonalized numerically or perturbatively in powers of $\lambda$. (3d Ising model done this way by R. Konik and Y. Adamov, 2007)

Real space renormalization group: we can study the dependence of physical quantities (mass gap, string tensions) on the truncation energy $E_n$.

# Questions?