Pseudoscalar flavor-singlet mixing angle and decay constants from $N_f = 2 + 1 + 1$ WtmLQCD

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Previous talk focused on masses; this talk on $\eta, \eta'$ mixing

- Lattice setup
- Definition of mixing parameters
- Mixing angle(s)
- Decay constants
- Quark mass dependence, extrapolations
- Decay widths $\Gamma(\eta \rightarrow \gamma\gamma), \Gamma(\eta' \rightarrow \gamma\gamma)$

Results for are still preliminary!
\( \eta, \eta' \) on the lattice

We work in the Wilson twisted mass \( N_f = 2 + 1 + 1 \) unitary setup:

\[
S_{F,l}(U, \chi_l, \bar{\chi}_l) = a^4 \sum_x \bar{\chi}_l \left( D_W + m_0 + i \mu_l \gamma_5 \tau^3 \right) \chi_l ,
\]

\[
S_{F,h}(U, \chi_h, \bar{\chi}_h) = a^4 \sum_x \bar{\chi}_h \left( D_W + m_0 + i \mu_\sigma \gamma_5 \tau^1 + \mu_\delta \tau^3 \right) \chi_h.
\]

Frezzotti et al., JHEP 0108:058 (2001)


- Automatic \( \mathcal{O}(a) \) improvement \( \rightarrow \mathcal{P} \) and \( \mathcal{F} \) at finite \( a \)
- Heavy sector not flavor-diagonal \( \rightarrow \) two additional propagators \( G_{cs}^{xy}, G_{sc}^{xy} \)

\( \Rightarrow \) Much more contractions for correlation functions in heavy sector

\( \Rightarrow \) Cannot apply tm variance reduction trick for heavy quarks

In the physical basis 2 \( \gamma \)-combinations \( (i \gamma_5, i \gamma_0 \gamma_5) \) available; consider only \( i \gamma_5 \):

phys basis:

\[
\eta_l^{\text{phys}} = \frac{1}{\sqrt{2}} \bar{\psi}_l i \gamma_5 \psi_l , \quad \eta_c^{\text{phys}} = \bar{\psi}_h \left( \frac{1 + \tau^3}{2} i \gamma_5 \right) \psi_h = \begin{pmatrix} \bar{c} i \gamma_5 c \\ \bar{s} i \gamma_5 s \end{pmatrix}
\]

tm basis:

\[
\eta_l^{\text{tm}} = \frac{1}{\sqrt{2}} \bar{\chi}_l (-\tau^3) \chi_l , \quad \eta_c^{\text{tm}} = \frac{1}{2} \bar{\chi}_h (-\tau^1 + i \gamma_5 \tau^3) \chi_h
\]

\( \Rightarrow \) heavy operators are a sum of scalars and pseudoscalars
Considering renormalization we have

\[
\begin{align*}
\eta_{c,\text{renormalized}}^{tm} &= Z \left( \bar{\chi}_c i \gamma_5 \chi_c - \bar{\chi}_s i \gamma_5 \chi_s \right) / 2 - (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s) / 2 \\
\eta_{s,\text{renormalized}}^{tm} &= Z \left( \bar{\chi}_s i \gamma_5 \chi_s - \bar{\chi}_c i \gamma_5 \chi_c \right) / 2 - (\bar{\chi}_s \chi_c + \bar{\chi}_c \chi_s) / 2 .
\end{align*}
\]

\[
\rightarrow \text{Need } Z = \frac{Z_P}{Z_S} ; \text{ can avoid this for masses ...}
\]

Additional rotation of basis to disentangle „heavy“ operators

\[
\eta_{S,P} = \eta_{c}^{tm} \pm \eta_{s}^{tm} = \left\{ \begin{array}{c}
\frac{1}{\sqrt{2}} \left( \bar{\chi}_c \chi_s + \bar{\chi}_s \chi_c \right) \\
\frac{1}{\sqrt{2}} \left( \bar{\chi}_c i \gamma_5 \chi_c - \bar{\chi}_s i \gamma_5 \chi_s \right)
\end{array} \right\} .
\]

In tm-basis we calculate:

\[
C^n(t) = \begin{pmatrix}
\eta_l(t)\eta_l(0) & \eta_l(t)\eta_S(0) & \eta_l(t)\eta_P(0) \\
\eta_S(t)\eta_l(0) & \eta_S(t)\eta_S(0) & \eta_S(t)\eta_P(0) \\
\eta_P(t)\eta_l(0) & \eta_P(t)\eta_S(0) & \eta_P(t)\eta_P(0)
\end{pmatrix} .
\]

Advantage: Number of contractions per matrix element reduced by a factor 4

Putting in \( Z \) and rotating back before solving GEVP:

\[
\Rightarrow \text{Eigenvectors of } C^n(t) \text{ give access to physical amplitudes } \rightarrow \text{mixing parameters}
\]
Mixing (I)

Decay constants are defined from axial-vector matrix elements (amplitudes)

$$\langle 0 | A^i_\mu | P(p) \rangle = i f^i_P p_\mu, \quad P = \eta, \eta',$$

either in singlet-octet ($i=0,8$) or quark flavor basis ($i=l,s$)

$$A^0_\mu = \frac{1}{\sqrt{6}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s), \quad A^l_\mu = \frac{1}{\sqrt{2}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d),$$

$$A^8_\mu = \frac{1}{\sqrt{3}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2\bar{s} \gamma_\mu \gamma_5 s), \quad A^s_\mu = \bar{s} \gamma_\mu \gamma_5 s.$$  

$\eta$ and $\eta'$ are not pure states in either basis; most general parametrization:

$$\begin{pmatrix} f^8,l & f^0,s \\ f^8,l & f^0,s \end{pmatrix} = \begin{pmatrix} f^8,l \cos \phi^8,l & -f^0,s \sin \phi^0,s \\ f^8,l \sin \phi^8,l & f^0,s \cos \phi^0,s \end{pmatrix}$$

From $\chi$PT one expects

- $|\phi^8 - \phi_0|$ is given by $SU(3)_F$ breaking terms; NOT small $|\phi^8 - \phi_0| \ll 1$
- $|\phi_l - \phi_s| \sim O(1/N_C)$ → small (?) OZI correction $|\phi_l - \phi_s| \ll 1$
Mixing (II)

On the lattice: quark flavor basis is “natural” choice

- Can check whether $|\phi_l - \phi_s|$ is small!
- Expect that only one angle $\phi \approx \phi_l \approx \phi_s$ is required:

$$\tan^2(\phi) = -\frac{f_l^{\eta'} f_s^{\eta}}{f_l^{\eta} f_s^{\eta'}}.$$ 

Singlet-octet and quark flavor angles are related

- $\phi_0 = \phi - \arctan(\sqrt{2} f_l/f_s) + O(1/N_C)$,
- $\phi_8 = \phi - \arctan(\sqrt{2} f_s/f_l) + O(1/N_C)$.

- In an SU(3)$_F$ symmetric world: “ideal” angle $\phi_{SU(3)_F} \approx 54.7^\circ$

- Small angle difference in one basis does NOT imply small difference in other basis!

Unfortunately, the axial vector is too noisy to determine $\phi/\phi_{l,s}$ and $f_{l,s}$ directly
Mixing (III)

Pseudoscalar amplitude

\[ h^i_P = 2m_i < 0 | P^i | P > , \quad P = \eta, \eta' , \]

is related to axial vector via the anomaly equation (singlet-octet)

\[ \partial^\mu A^i_\mu = \bar{\psi}(x) 2MT^i \gamma_5 \psi(x) + \delta^{i0} \sqrt{2N_f} \omega(x) . \]

In the quark flavor basis this leads to

\[ \left( \begin{array}{cc} P_{l,\eta} & P_{s,\eta} \\ P_{l,\eta'} & P_{s,\eta'} \end{array} \right) = \left( \begin{array}{cc} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{array} \right) \text{diag}(f_l M_{PS}^2, f_s (2M_K^2 - M_{PS}^2)) . \]

- This expression holds to LO \( \chiPT \)
- Ignoring higher orders in \( O(1/N_C) \) (i.e. \( \phi_l \approx \phi_s \)) AND higher orders in masses

\[ \rightarrow \text{some } \chiPT\text{-dependence compared to axial-vector approach} \]
Two-photon decay widths

$\eta, \eta'$-mixing parameters are related to anomaly $\rightarrow$ relevance for several processes

- The decays $\eta, \eta' \rightarrow \gamma\gamma$ are driven by the chiral anomaly

- At LO: Wess-Zumino-Wittten term

$$L_{WZW}^{\text{LO}} = - \frac{N_C \alpha_{\text{QED}}}{4\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \text{tr}\left[\text{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) \phi^2\right].$$

Tree level prediction for decay widths reads

$$\Gamma [\eta \rightarrow \gamma\gamma] = \frac{\alpha_{\text{QED}}^2}{576\pi^3} M_\eta^3 \left[ \frac{5}{f_l} \cos \phi - \frac{\sqrt{2}}{f_s} \sin \phi \right]^2,$$

$$\Gamma [\eta' \rightarrow \gamma\gamma] = \frac{\alpha_{\text{QED}}^2}{576\pi^3} M_{\eta'}^3 \left[ \frac{5}{f_l} \sin \phi + \frac{\sqrt{2}}{f_s} \cos \phi \right]^2.$$


- OZI-suppressed terms are dropped $\rightarrow$ consistent with mixing scheme

- Expressions become rigorous in the chiral limit
Setup

- We use almost all ETMC $N_f = 2 + 1 + 1$ ensembles (16 ensembles)
- Three lattice spacings $a_A = 0.086\, \text{fm}$, $a_B = 0.078\, \text{fm}$ and $a_D = 0.061\, \text{fm}$
- Physical lattice size $L \geq 3\, \text{fm}$ for many ensembles; $LM_{\text{PS}} \geq 3.5$
- $\sim 600$ up to $\sim 2500$ gauge configuration per ensemble
- Charged pion masses range from $\sim 230\, \text{MeV}$ to $\sim 500\, \text{MeV}$
- Bare $m_s$, $m_c$ fixed for each $\beta$
- We remove excited contributions in conn correlators $\rightarrow$ previous talk
Linear fit: $\phi = 46.0^\circ (0.9)_{\text{stat}} (2.7)_{\text{sys}}$

Systematic error parametrizes ignorance towards $m_s, a$-dependence

Compatible with old analysis $\phi = 44^\circ (5)_{\text{stat}}$ and other lattice and experimental results. 

Ottnad et. al., JHEP 1211 (2012) 048
Linear fits: $\phi_l = 47.7^\circ (1.2)_{\text{stat}} (4.1)_{\text{sys}}$ and $\phi_s = 44.3^\circ (0.9)_{\text{stat}} (3.0)_{\text{sys}}$

Difference $\Delta \phi_{ls} = 2.8^\circ (1.1)_{\text{stat}} (2.6)_{\text{sys}}$ confirms smallness of OZI-corrections

$\Rightarrow$ data well described by single angle in quark flavor basis
Decay constants - $f_i$

- $f_i$ shows rather nonlinear $m_l$-dependence; scaling artifacts
- Most $m_l$-dependence cancels in the ratio $f_i/f_{PS}$
- $m_s$-dependence negligible
- Linear fit: $f_i/f_{PS} = 0.859(7)_{\text{stat}}(64)_{\text{sys}}$
- Fit to finest lattice spacing only $f_i/f_{PS}\vert_D = 0.924(22)_{\text{stat}}$
- Phenomenology $f_i/f_{\pi} = 1.07(2)$

Decay constants - $f_s$

- $f_s$ shows sizable $m_s$-dependence; possibly scaling artifacts
- $f_s/f_K$ cancels most $m_s, a$-dependence; rather mild $m_l$-dependence
- Linear fit: $f_s/f_K = 1.166(11)_{\text{stat}}(31)_{\text{sys}}$
- Phenomenology $f_s/f_K = 1.12(6)$

Two-photon decay widths

- Very preliminary; formulae used for decay widths are tree level only
- $\Gamma[\eta \to \gamma\gamma]$ shows nonlinear $m_l$-dependence; additional $m_s$-dependence?
- $\Gamma[\eta' \to \gamma\gamma]$ rather compatible exp. value; still some $a$-dependence (possibly also $m_s$-dependence...)

Need better control of scaling artifacts and $m_q$-dependence for definite results!
Summary and Outlook

- First lattice determination of $\eta, \eta'$ decay constants

  - (Preliminary) results:
    \[
    \phi = 46.0^{\circ}(0.9)_{\text{stat}}(2.7)_{\text{sys}}, \quad \phi_{\text{phenom}} = 39.3^{\circ}(1.0)
    \]
    \[
    f_l/f_{PS} = 0.859(07)_{\text{stat}}(64)_{\text{sys}}, \quad (f_l/f_{PS})_{\text{phenom}} = 1.07(2)
    \]
    \[
    f_s/f_K = 1.166(11)_{\text{stat}}(31)_{\text{sys}}, \quad (f_s/f_K)_{\text{phenom}} = 1.12(6)
    \]

- Determination of $\phi, f_s/f_K$ with controlled systematics
- Our study confirms smallness of OZI corrections in quark flavor basis
- Still need better control of lattice artifacts for $f_l/f_{PS}$
- Decay widths for $\eta, \eta' \rightarrow 2\gamma$ accessible; need control of systematics

Further plans:

- Vary $m_s$ for further ensembles
- Point-to-point correlators...
- ... maybe get signal for axial vector $\rightarrow$ direct access to mixing parameters
Removal of excited states (I)

**Problem:** (Large) disconnected contributions to $\eta'$

- Signal for $\eta'$ lost at small $t$
- Hardly plateau for $M_{\eta'}$; impossible to extract amplitudes
- Large contamination from excited states

**Possible solutions:**

- Use much larger statistics $\rightarrow$ very expensive
- Increase operator basis $\rightarrow$ not easily possible, axial vector very noisy
- Point-to-point correlators; stoch. distillation $\rightarrow$ will be tested
- … or find some other method to extract quantities at small $t$
Removal of excited states (II)

- Ignore charm quark
- Consider $\mathcal{M}^2 = \text{diag}(M^2_\eta, M^2_{\eta'})$ in quark flavor basis

$$
\mathcal{M}^2 = \begin{pmatrix}
M^2_{ll} + 2\Delta_{ll} & \sqrt{2}\Delta_{ls} \\
\sqrt{2}\Delta_{ls} & M^2_{ss} + \Delta_{ss}
\end{pmatrix}
$$

- $M_{ll}, M_{ss}$: masses of flavor non-singlet eigenstates (connected only)
- $\Delta_{ll}, \Delta_{ls}$ and $\Delta_{ss}$ give large corrections (disconnected)

Assumption:

Disconnected diagrams couple only to $\eta, \eta'$

- Replace connected contributions by respective ground state contributions
- If assumption is correct we should see a plateau at very low $t$
Comparison of results using improved method
Two-photon decay widths (II)