Relativistic Bose gas on a Lefschetz thimble

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Introduction: Lefschetz thimble

Path integral and Morse theory

- Complexify the degrees of freedom

\[ \int_{\mathbb{R}^n} dx^n g(x) e^{f(x)} \rightarrow \int_{\mathcal{C}} dz^n g(z) e^{f(z)} \]

- Deform appropriately the original integration path (Morse theory)

\[ \int_{\mathcal{C}} dz^n g(z) e^{f(z)} = \sum_{\sigma} n_{\sigma} \int_{\mathcal{L}_{\sigma}} dz^n g(z) e^{f(z)} \]

\( \mathcal{L}_{\sigma} \) for each stationary point \( p_{\sigma} \) the \( \mathcal{L}_{\sigma} \) (thimble) is the union of the paths of steepest descent that fall in \( p_{\sigma} \) at \( \infty \)

\[ \mathcal{C} = \sum_{\sigma} n_{\sigma} \mathcal{L}_{\sigma} \] the thimbles provide a basis of the relevant homology group, with integer coefficients

On the manifold defined by steepest descent the phase is fixed: no more oscillating integrals!!
Integration on a Lefschetz thimble
M. C., F. Di Renzo and L. Scorzato
PRD86, 074506 (2012)

Is it numerically applicable to QFT’s on a Lattice?

Before applying the idea to full QCD we choose to start from something more manageable: we consider here integration on the Lefschetz thimbles for the case of

- a 0 dimensional field theory with U(1) symmetry
- the four dimensional scalar field with a quartic interaction
Can be seen as the limiting case of the more interesting three-dimensional XY model

One dimensional problem: the integration on the Lefschetz thimble can be plotted

**ACTION**

\[ S = -i \frac{\beta}{2} (U + U^{-1}) = -i \beta \cos \phi \]

**OBSERVABLE**

\[ \langle e^{i\phi} \rangle = i \frac{J_1(\beta)}{J_0(\beta)} \]

On the thimble

\[ \langle \mathcal{O}(\phi) \rangle = \frac{\sum_{\sigma} m_{\sigma} \int_{\mathcal{J}_{\sigma}} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_{\sigma} m_{\sigma} \int_{\mathcal{J}_{\sigma}} d\phi e^{-S(\phi)}} \]

\[ S_R = -\beta \sin \phi_R \sinh \phi_I \]

\[ S_I = -\beta \cos \phi_R \cosh \phi_I \]

constant on the thimble
The stationary points are in (0,0) and (π,0) and the thimble can be computed also analytically.

Exact thimbles: have to pass from the critical point and the imaginary part of the action has to be constant.

\[ S_I(\tau) = -\beta \cos \phi_R(\tau) \cosh \phi_I(\tau) = S_I^{cp} \]
The stationary points are in \((0,0)\) and \((\pi,0)\) and the thimble can be computed also analytically.

In order to perform the integration on the thimble we use a Metropolis algorithm:

\[
\frac{d\phi}{d\tau} = -\frac{\partial S}{\partial \phi} \\
\phi' = \phi - \delta \tau \frac{\partial S}{\partial \phi}
\]

Gaussian approximation: flat manifold defined by the directions of steepest descent in the critical point.
The stationary points are in (0, 0) and (\(\pi, 0\)) and the thimble can be computed also analytically.

In order to perform the integration on the thimble we use a Metropolis algorithm.

Increasing the accuracy in the integration of the steepest descent we move closer to the exact thimble.
U(1) one plaquette model


OBSERVABLE

\[ \langle e^{i\phi} \rangle = i \frac{J_1(\beta)}{J_0(\beta)} \]

There are parameter regions where integration on the Gaussian manifold is sufficiently accurate.
Residual phase is well under control and is not a source of additional sign problem (at least in this case)

\[
\langle \mathcal{O}(\phi) \rangle = \frac{\sum_{\sigma} m_{\sigma} \int_{\mathcal{J}_{\sigma}} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_{\sigma} m_{\sigma} \int_{\mathcal{J}_{\sigma}} d\phi e^{-S(\phi)}}
\]

There is an additional phase coming from the Jacobian of the transformation between the canonical complex basis and the tangent space to the thimble.

This phase should be essentially constant over the portion of phase space which dominates the integral. But we have no formal argument.
\[ S[\phi, \phi^*] = \int d^4x \left( |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^* \partial_0 \phi - \partial_0 \phi^* \phi) \right) \]

Continuum action

\[ S^*[\mu] = S[-\mu^*] \]

Silver Blaze problem
when \( T=0 \) and \( \mu < \mu_c \) physics is independent from the chemical potential

We will study the system at zero temperature
\[ \lambda \Phi^4 \text{ theory on the lattice} \]

\[
S[\phi, \phi^*] = \int d^4x \left( |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^* \partial_0 \phi - \partial_0 \phi^* \phi) \right)
\]

Continuum action

\[
S[\phi, \phi^*] = \sum_x \left( (2d + m^2)\phi_x^* \phi_x + \lambda (\phi_x^* \phi_x)^2 \right)
\]

\[- \sum_{\nu=0}^{4} \left( \phi_x^* e^{-\mu \delta_{\nu,0}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu \delta_{\nu,0}} \phi_x \right) \]

Lattice action:
chemical potential introduced as an imaginary constant vector potential in the temporal direction

in term of real fields \( \phi_a (a = 1, 2) \) \( \phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \)

\[
S[\phi_a] = \sum_x \left[ \frac{1}{2} (2d + m^2) \phi_{a,x}^2 + \frac{\lambda}{4} (\phi_{a,x}^2)^2 - \sum_{\nu=1}^{3} \phi_{a,x} \phi_{a,x+\hat{\nu}} \right]
\]

\[- \cosh \mu \phi_{a,x} \phi_{a,x+\hat{0}} + i \sinh \mu \varepsilon_{ab} \phi_{a,x} \phi_{b,x+\hat{0}} \]
On the Lefschetz thimble
M. C., F. Di Renzo, A. Mukherjee and L. Scorzato

☐ Fields are complexified \( \phi_a \rightarrow \phi_a^R + i\phi_a^I \)

☐ The integration on the thimble performed with a Langevin algorithm

☐ In this case calculations in Gaussian approximation are sufficient to obtain the exact result
On the Lefschetz thimble
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Silver Blaze
solving sign problem we have the correct physics
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\[ \langle n \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial \mu} \]

Comparison with Worm Algorithm
(courtesy of C. Gattringer and T. Kloiber)
Something else on a Lefschetz thimble

Next steps

- XY Model
- Hubbard model (involves a determinant)
- ...
- move to QCD
thank you