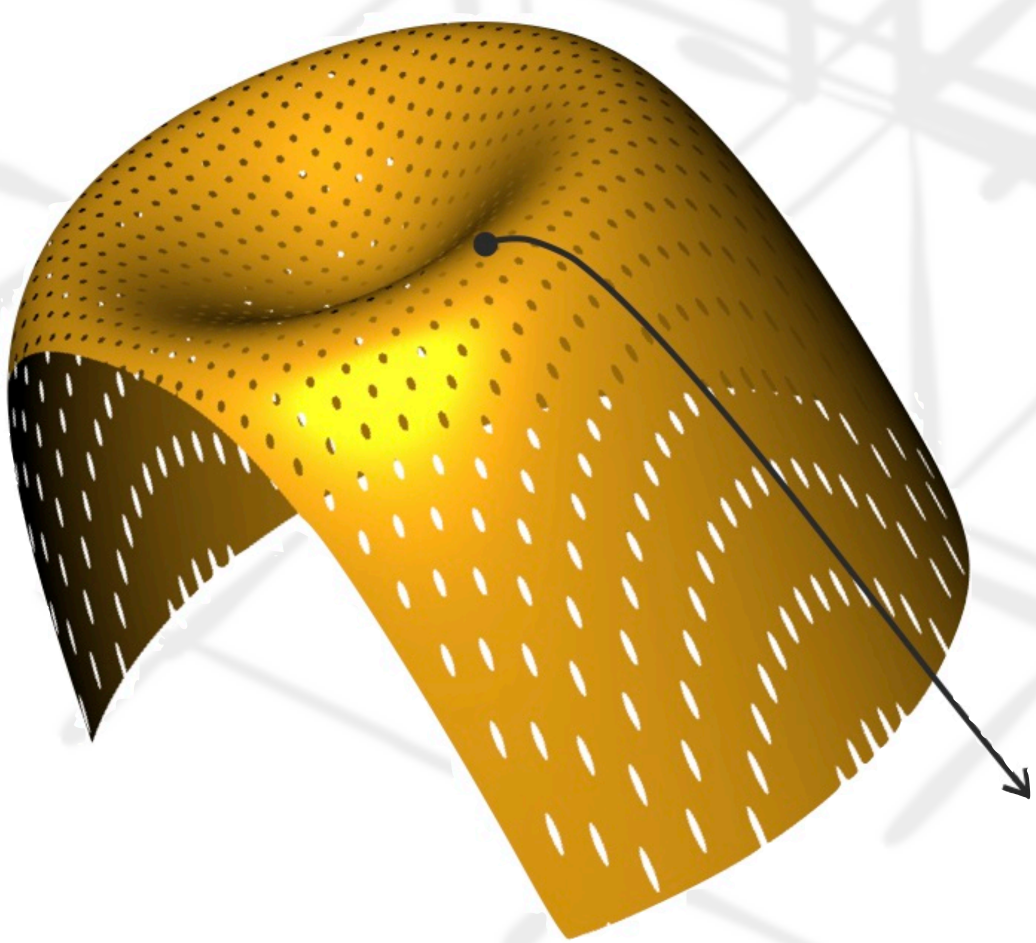
Relativistic Bose gas on a Lefschetz thimble

Marco Cristoforetti
ECT*

Luigi Scorzato
LISC

Francesco Di Renzo
University of Parma

Abhishek Mukherjee
ECT*



PRD86, 074506 (2012)
arXiv:1303.7204 (2013)
arXiv:1308.0233 (2013)

LATTICE
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Introduction: Lefschetz thimble

Path integral and Morse theory

E. Witten arXiv:1009.6032 (2010)

- Complexify the degrees of freedom

$$\int_{\mathbb{R}^n} dx^n\, g(x) e^{f(x)} \xrightarrow{z = x + iy} \int_{\mathcal{C}} dz^n\, g(z) e^{f(z)}$$

- Deform appropriately the original integration path (Morse theory)

$$\int_{\mathcal{C}} dz^n\, g(z) e^{f(z)} = \sum_\sigma n_\sigma \int_{\mathcal{L}_\sigma} dz^n g(z) e^{f(z)}$$

$\mathcal{L}_\sigma$ for each stationary point p_σ the L_σ (thimble) is the union of the paths of steepest descent that fall in p_σ at ∞

$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{L}_\sigma$ the thimbles provide a basis of the relevant homology group, with integer coefficients

On the manifold defined by steepest descent the phase is fixed: no more oscillating integrals !!

Introduction: Lefschetz thimble

Integration on a Lefschetz thimble

M. C., F. Di Renzo and L. Scorzato
PRD86, 074506 (2012)

Is it numerically applicable to QFT's on a Lattice?

Before applying the idea to full QCD we choose to start from something more manageable: we consider here integration on the Lefschetz thimbles for the case of

→ **a 0 dimensional field theory with U(1) symmetry**

→ **the four dimensional scalar field with a quartic interaction**

U(1) one plaquette model

A. Mukherjee, M. C. and L. Scorzato arXiv:1308.0233 (2013)

- Can be seen as the limiting case of the more interesting three-dimensional XY model
- One dimensional problem: the integration on the Lefschetz thimble can be plotted

ACTION $$S = -i\frac{\beta}{2}\left(U + U^{-}1\right) = -i\beta\cos\phi$$

OBSERVABLE $$\langle e^{i\phi}\rangle = i\frac{J_1(\beta)}{J_0(\beta)}$$

On the thimble

$$\langle \mathcal{O}(\phi)\rangle = \frac{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi (\phi) e^{-S(\phi)}}$$

$$S_R = -\beta\sin\phi_R\sinh\phi_I$$

$$S_I = -\beta\cos\phi_R\cosh\phi_I$$

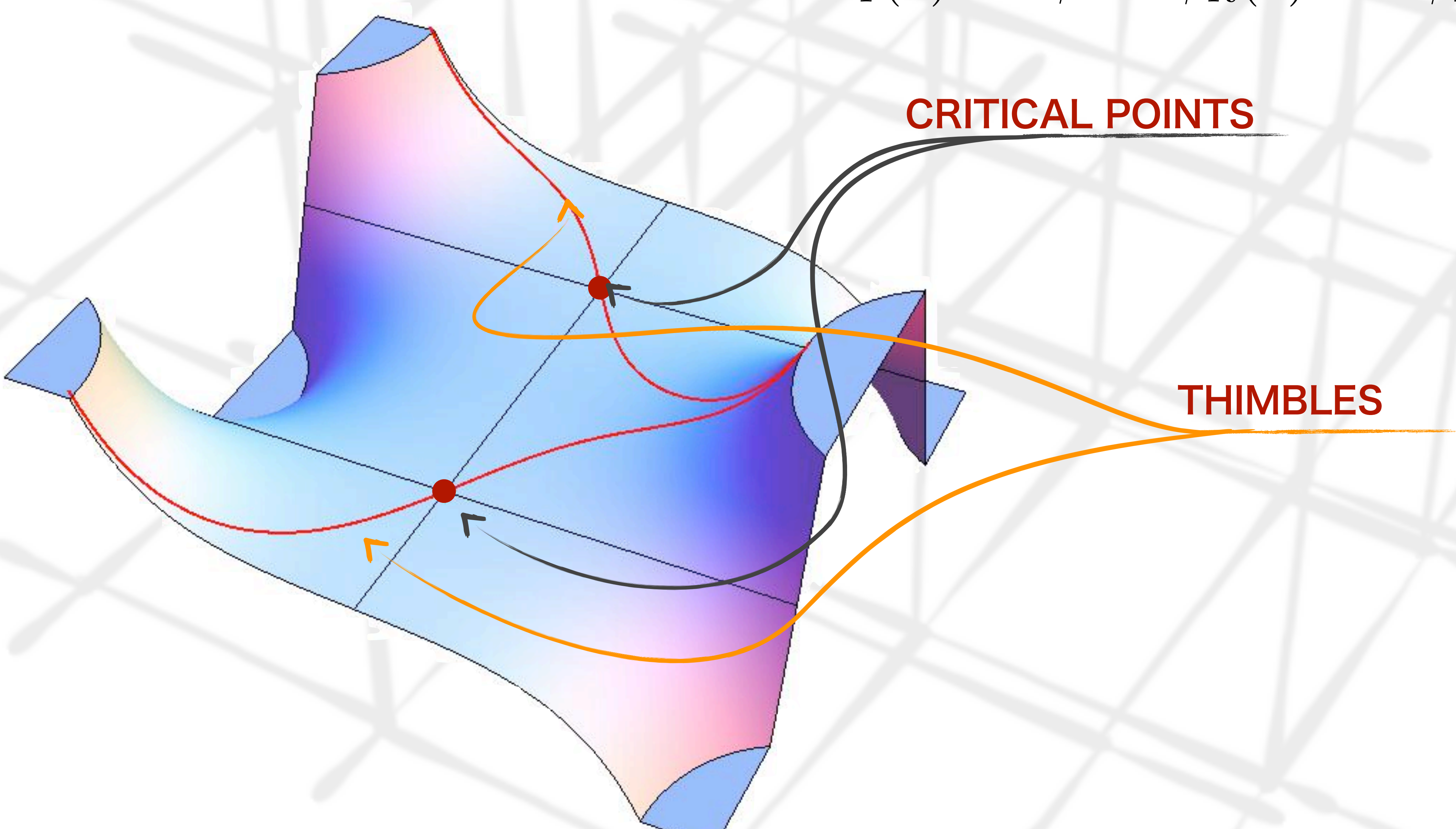
constant on the thimble

U(1) one plaquette model

A. Mukherjee, M. C. and L. Scorzato arXiv:1308.0233 (2013)

- The stationary points are in (0,0) and (π,0) and the thimble can be computed also analytically
  - Exact thimbles: have to pass from the critical point and the imaginary part of the action has to be constant

$$S_I(\tau) = -\beta \cos\phi_R(\tau)\cosh\phi_I(\tau) = S_I^{\mathrm{cp}}$$



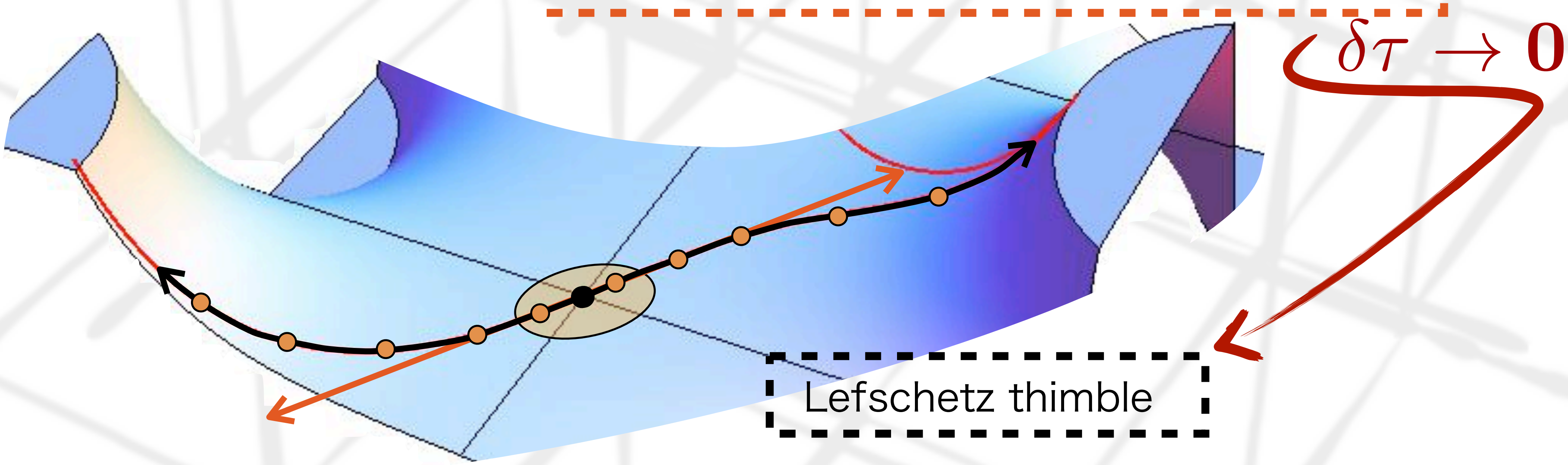
U(1) one plaquette model

A. Mukherjee, M. C. and L. Scorzato arXiv:1308.0233 (2013)

- The stationary points are in (0,0) and (π,0) and the thimble can be computed also analytically
  - In order to perform the integration on the thimble we use a Metropolis algorithm

$$\frac{d\phi}{d\tau} = -\overline{\frac{\partial S}{\partial \phi}} \quad \cdots\cdots\rightarrow \quad \phi' = \phi - \delta\tau \overline{\frac{\partial S}{\partial \phi}}$$

Gaussian approximation:
flat manifold defined by the directions of steepest descent in the critical point



U(1) one plaquette model

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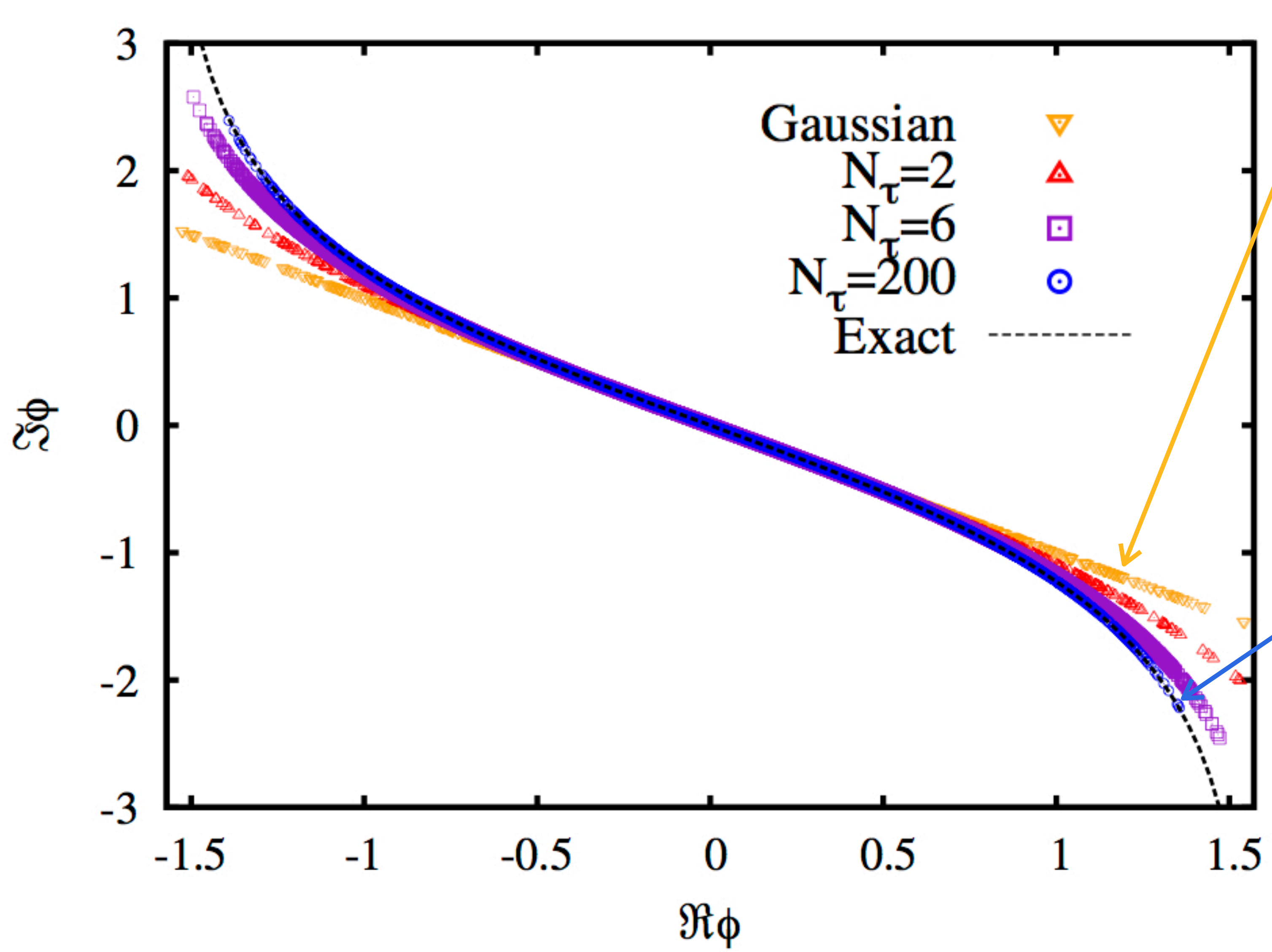
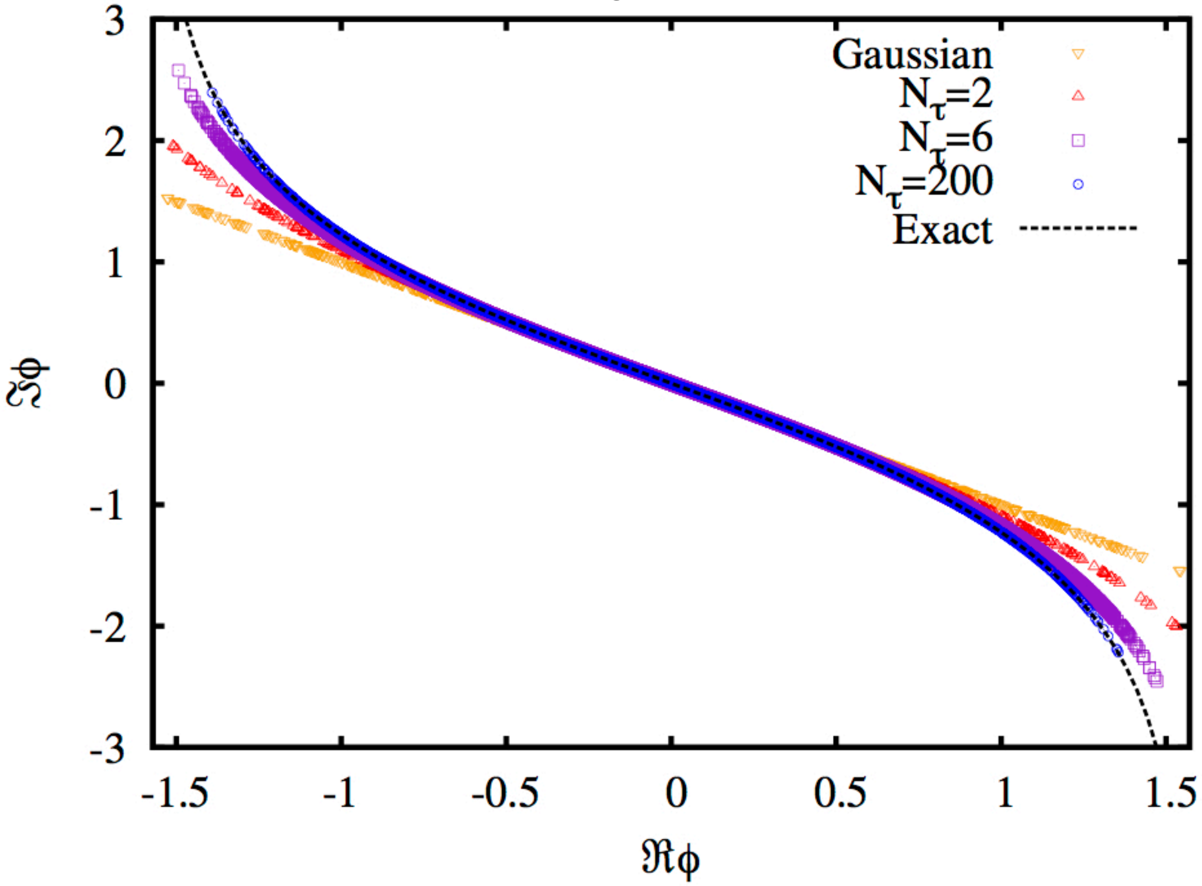


Gaussian manifold

Increasing the accuracy in the integration of the steepest descent we move closer to the exact thimble

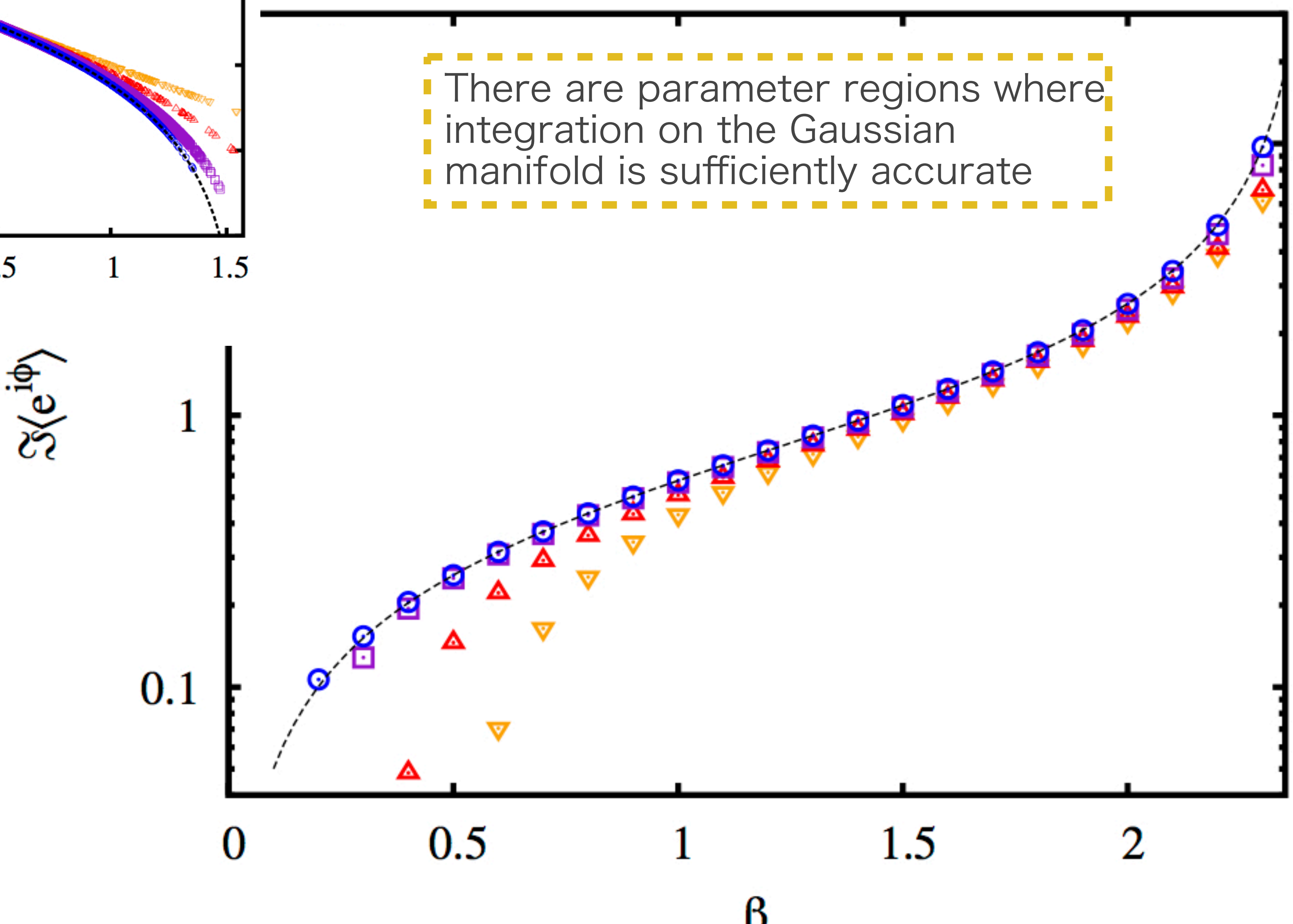
U(1) one plaquette model

A. Mukherjee, M. C. and L. Scorzato arXiv:1308.0233 (2013)



OBSERVABLE

$$\langle e^{i\phi}\rangle = i\frac{J_1(\beta)}{J_0(\beta)}$$



U(1) one plaquette model

A. Mukherjee, M. C. and L. Scorzato arXiv:1308.0233 (2013)

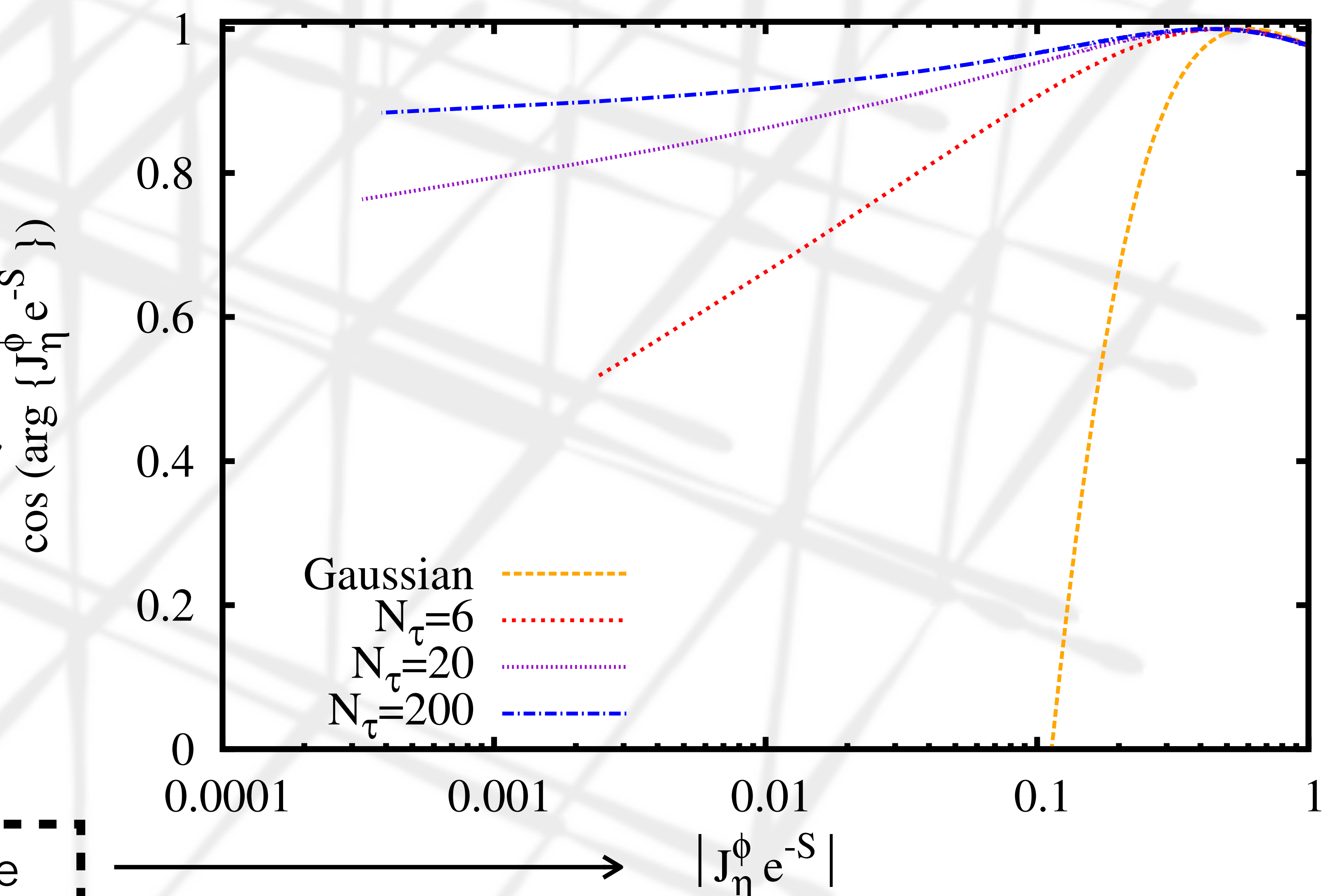
- Residual phase is well under control and is not a source of additional sign problem (at least in this case)

$$\langle \mathcal{O}(\phi) \rangle = \frac{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi \mathcal{O}(\phi) e^{-S(\phi)}}{\sum_\sigma m_\sigma \int_{\mathcal{J}_\sigma} d\phi (\phi) e^{-S(\phi)}}$$

There is an additional phase coming from the Jacobian of the transformation between the canonical complex basis and the tangent space to the thimble

This phase should be essentially constant over the portion of phase space which dominates the integral. But we have no formal argument

Residual phase



Probability measure

$\lambda\,\Phi^4$ theory on the lattice

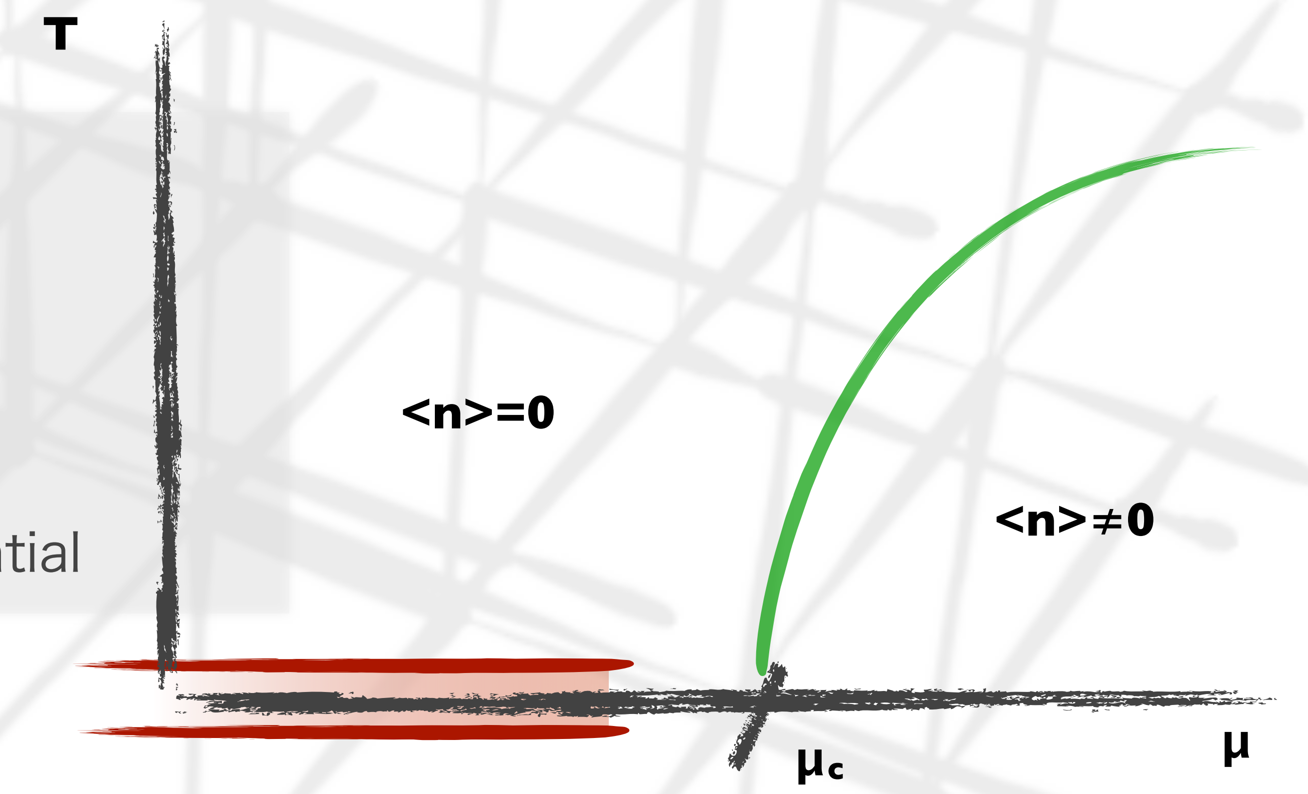
$$S[\phi,\phi^*] = \int d^4x\left(|\partial_\nu\phi|^2 + (m^2-\mu^2)|\phi|^2 + \lambda|\phi|^4 + \mu(\phi^*\partial_0\phi - \partial_0\phi^*\phi)\right)$$

Continuum action

$S^*[\mu]=S[-\mu^*]$

Silver Blaze problem

when T=0 and $\mu<\mu_c$ physics is independent from the chemical potential



We will study the system at zero temperature

$\lambda\,\Phi^4$ theory on the lattice

$$S[\phi,\phi^*]=\int d^4x(|\partial_\nu\phi|^2+(m^2-\mu^2)|\phi|^2+\lambda|\phi|^4+\mu(\phi^*\partial_0\phi-\partial_0\phi^*\phi)$$

Continuum action

$$S[\phi,\phi^*]=\sum_x[(2d+m^2)\phi_x^*\phi_x+\lambda(\phi_x^*\phi_x)^2$$
$$-\sum_{\nu=0}^{4}(\phi_x^*e^{-\mu\delta_{\nu,0}}\phi_{x+\hat\nu}+\phi_{x+\hat\nu}^*e^{\mu\delta_{\nu,0}}\phi_x))$$

Lattice action:
chemical potential introduced as an imaginary constant vector potential in the temporal direction

in term of real fields $\phi_a(a=1,2)$ $\quad \phi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$

$$S[\phi_a]=\sum_x\left[\frac{1}{2}(2d+m^2)\phi_{a,x}^2+\frac{\lambda}{4}(\phi_{a,x}^2)^2-\sum_{\nu=1}^{3}\phi_{a,x}\phi_{a,x+\hat{i}}\right.$$
$$\left.-\cosh\mu\,\phi_{a,x}\phi_{a,x+\hat0}+i\sinh\mu\,\varepsilon_{ab}\phi_{a,x}\phi_{b,x+\hat0}\right]$$

$\lambda\,\Phi^4$ on a Lefschetz thimble

On the Lefschetz thimble

M. C., F. Di Renzo, A. Mukherjee and L. Scorzato
arXiv:1303.7204 (2013)

- Fields are complexified $\phi_a \to \phi_a^R + i\phi_a^I$
- The integration on the thimble performed with a Langevin algorithm
- In this case calculations in Gaussian approximation are sufficient to obtain the exact result

$\lambda\,\Phi^4$ on a Lefschetz thimble

On the Lefschetz thimble

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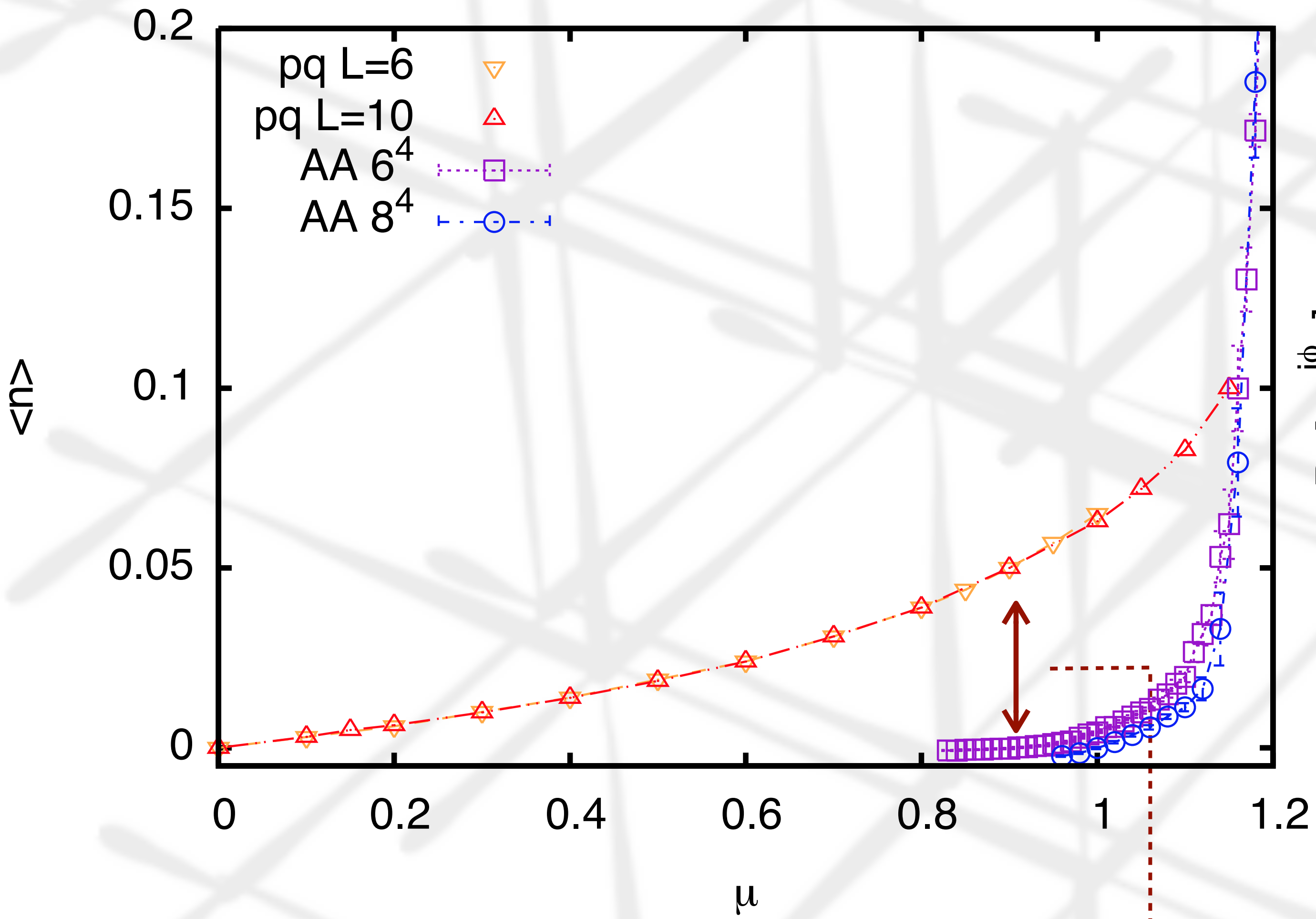

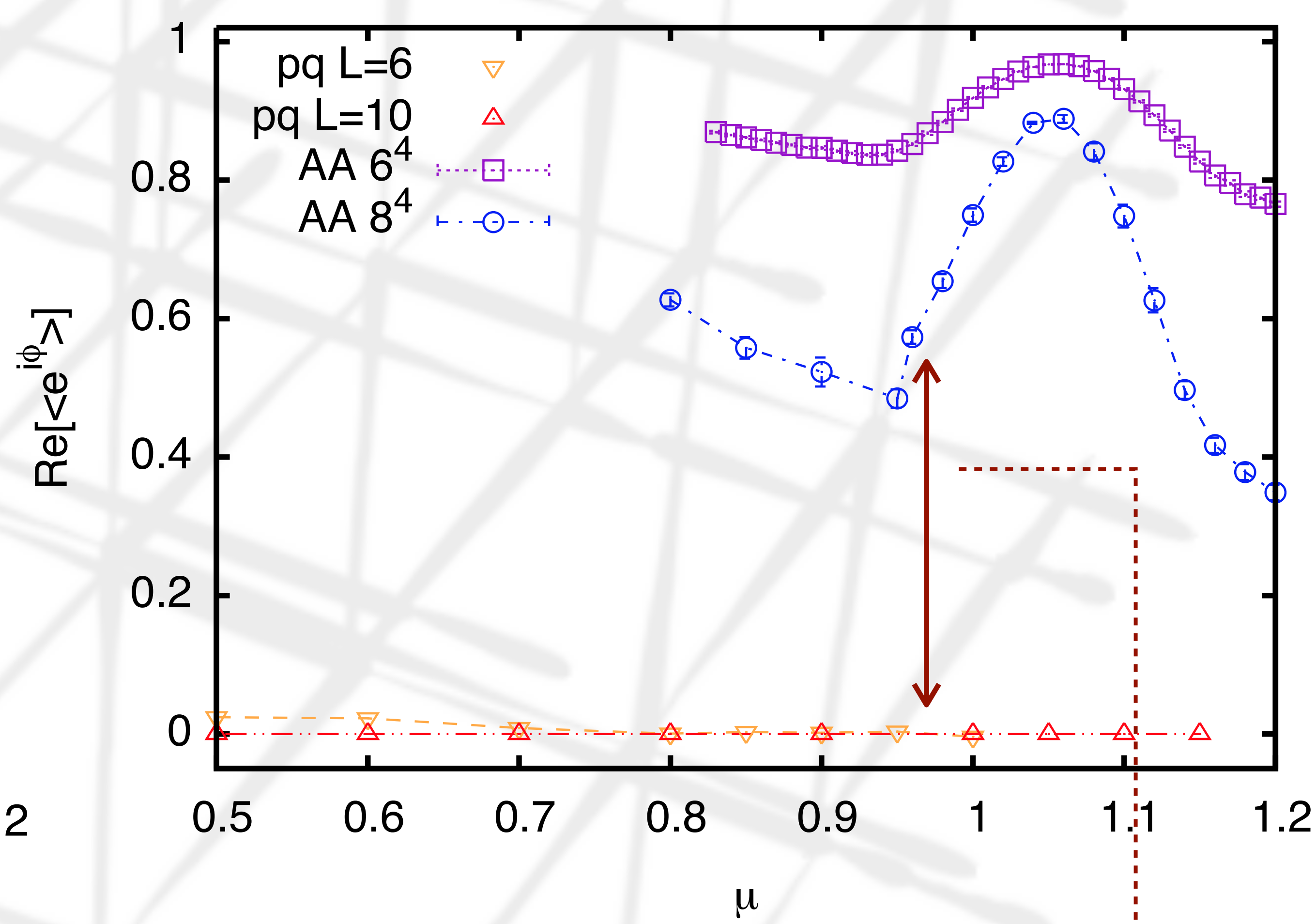

Silver Blaze

solving sign problem we have the correct physics

$\lambda\,\Phi^4$ on a Lefschetz thimble

On the Lefschetz thimble

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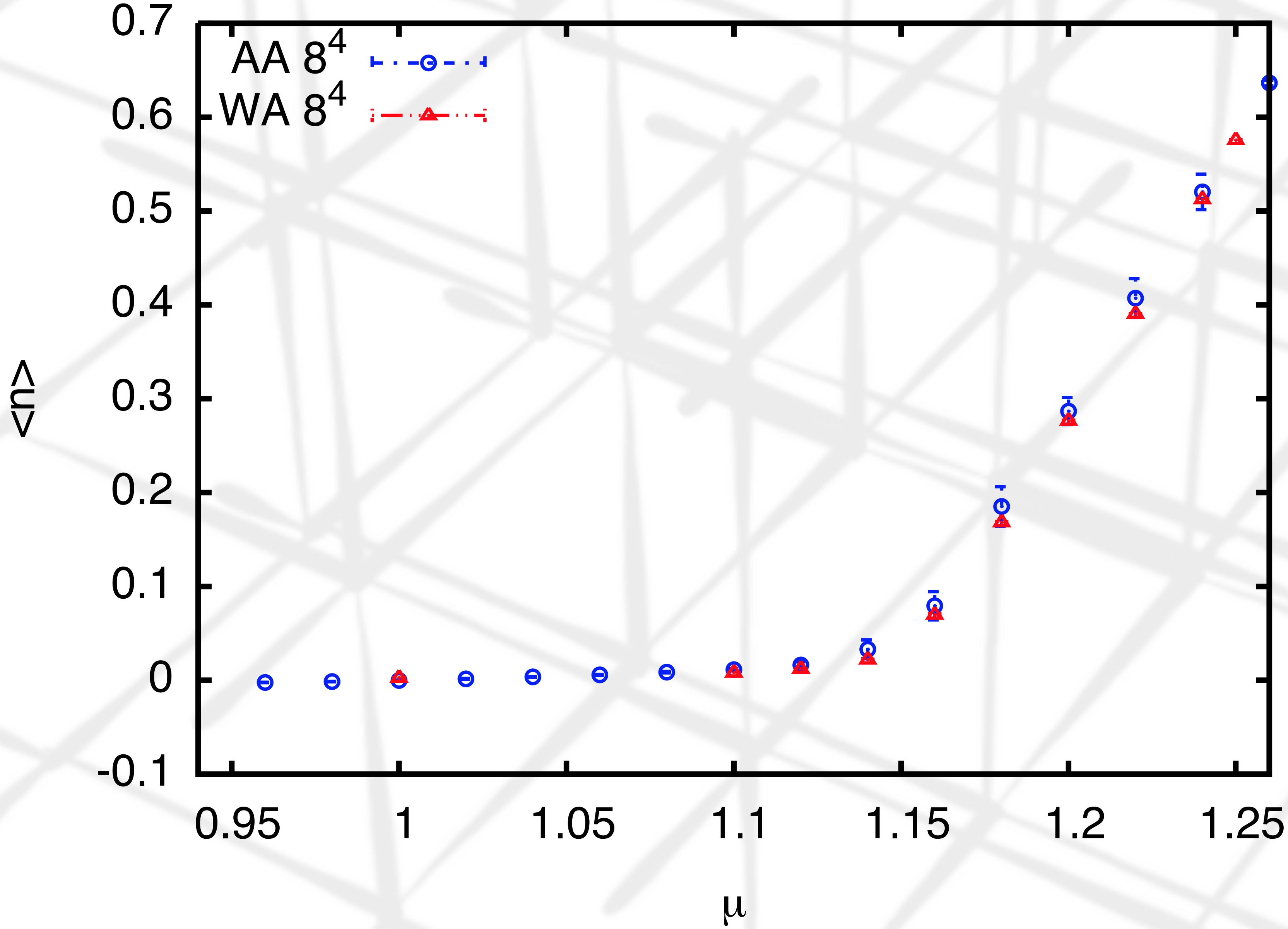

Comparison with Worm Algorithm
(courtesy of C. Gattringer and T. Kloiber)

$$\langle n \rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$$

Something else on a Lefschetz thimble

Next steps

- XY Model
- Hubbard model (involves a determinant)
- ...
- move to QCD

thank **you**