Towards a density of states approach for dense matter systems

Lattice 2013, Mainz

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Finite densities of heavy quarks

- SU(N) gauge theories at finite densities of heavy quarks:

  △ start: continuum quark determinant with μ

  △ systematic expansion in $1/m$ (heat kernel expansion)
  [Langfeld, Shin, Nucl.Phys. B572, 266 (2000)]

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- result for $\mu \lesssim m$:

  $$S[U] = S_{\text{pure}}[U] \; + \; f \; p[U], \qquad p[u] := \sum_{\vec{x}} P(\vec{x})$$

  $P(\vec{x})$: (traced) Polyakov line, $f = \sqrt{2}\pi^{-3/2}(mT)^{3/2}a^3 \; \exp\{(\mu - m)/T\}$

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- "weak coupling" version of the Polyakov line spin model
  No silver blaze problem!
  small densities only (no saturation on the lattice)

Finite densities of heavy quarks

- Quantities of interest - effective potential

$$\boxed{V(q) = \frac{T}{V_3}(j\,q - \ln Z[J])}\,, \qquad q = \frac{d\,\ln Z[j]}{dj} = \langle p[U] \rangle .$$

$$Z[j] = \int \mathcal{D}U_\mu \; \exp\Big\{ S_{\rm pure}[U] + j \sum_{\vec{x}} P(\vec{x}) \Big\}$$

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- Challenges:

$\triangle$ poor statistics: 1 configuration $\Rightarrow 1\ p[U]$

$\triangle$ poor signal-to-noise ratio: "$j\sum_{\vec{x}} P(\vec{x})$" cancles "$jq$"

$\triangle$ overlap problem!

$\triangle$ SU($N > 2$) (weak?) sign problem!

The density-of-states method (LLR)

We need a numerical method to calculate $Z[j]$ with

exponential error suppression

for a wide range of j !

The LLR approach

- What is the density of states? `[my definition]`
  start with a partition function: $Z = \int \mathcal{D}\phi \, \exp\{\beta S[\phi]\}$

  β : coupling constant (QFT), inverse temperature (solid state physics)

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- What can we do with $\rho(E)$?

  Get the partition function for all β: $Z \;=\; \int dE \, \rho(E) \, e^{\beta E}$

  `[scaling analysis in QFT]`

  `Directly` access the free energy

  $\Rightarrow$ thermal energy density, pressure, latent heat (1st order transitions), interface tensions, ...

The LLR approach ($\mu = 0$)

- How do we calculate $\rho(E)$ numerically?

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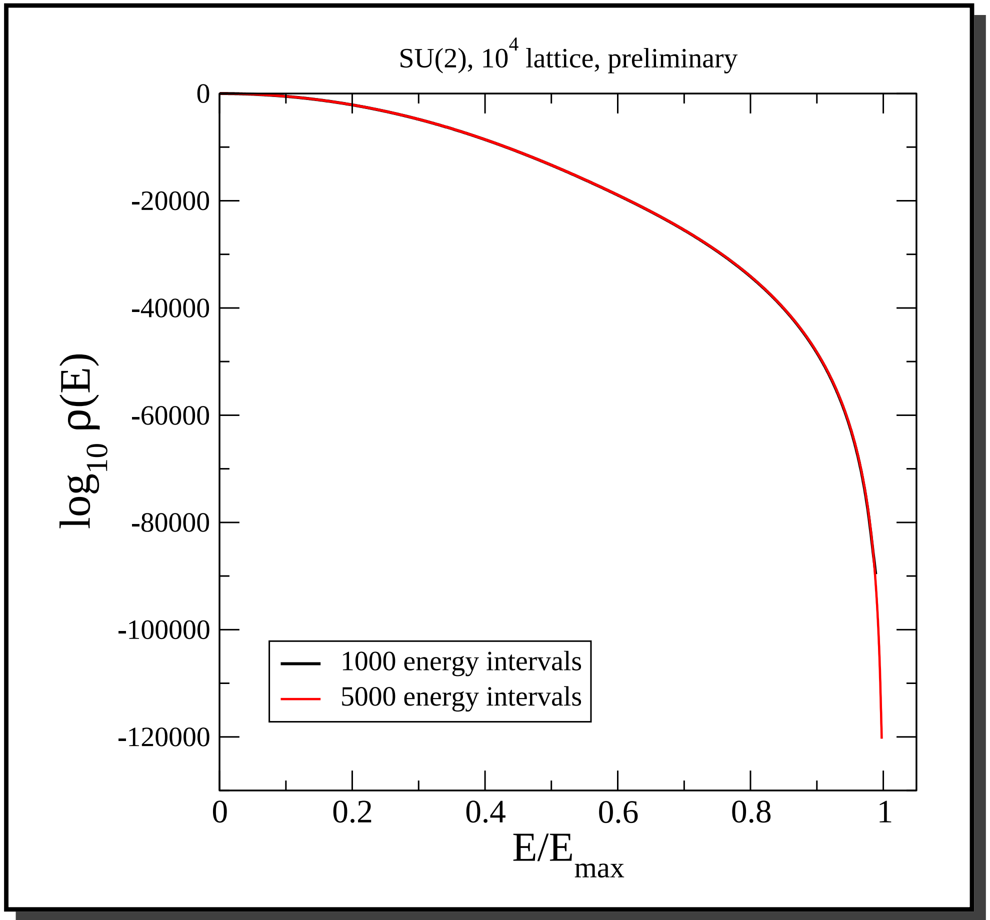
- How do we calculate $\rho(E)$ numerically?
- Observation:

  $\log \rho(E)$ is a *remarkable* smooth function of E!

example:

SU(2), 10^4 lattice:

$E_{\text{max}} = 60,000$:



The LLR approach

- Choose a piecewiese linear ansatz:

$$\rho(E) = \rho(E_0)\, \exp\Big\{ a(E_0)\, (E - E_0) \Big\}, \quad E_0 < E < E_0 + \delta E$$

need to find the $a(E_0)$!

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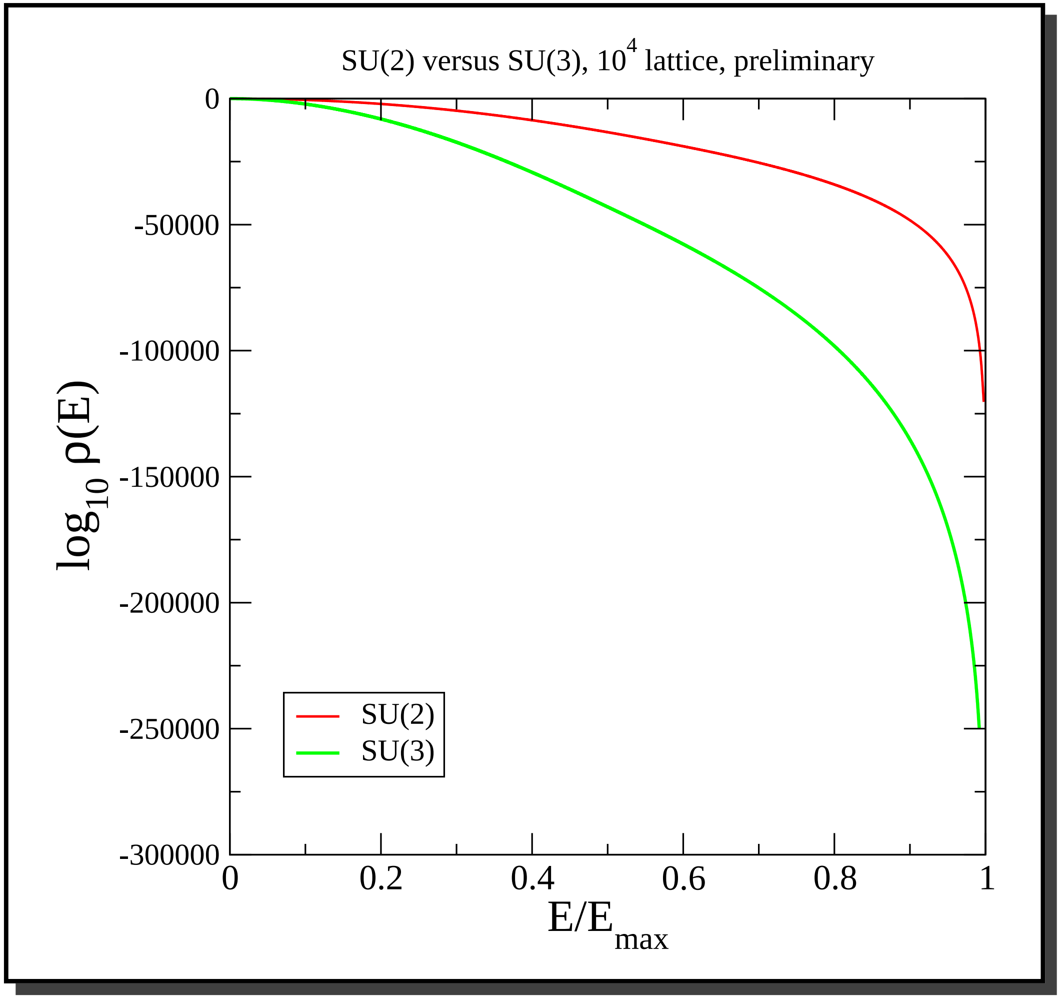
- The LLR algorithm:
  △ derive non-linear equation for $a(E_0)$
  △ uses MC expectation values (truncation + reweighting)
  △ use Newton-Raphson to find $a(E_0)$

[Langfeld, Lucini, Rago, Phys.Rev.Lett. 109 (2012) 111601]

The density of states - SU(3) versus SU(2)

- Results for SU(3) versus SU(2):



The density of states - compact U(1)

- study phase transition in U(1):

$\triangle$ weakly first order
[Arnold, Bunk, Lippert, Schilling, Nucl.Proc.Proc.Suppl. 119 (2003) 119]

$\triangle$ $P_\beta(E) = \rho(E) \exp\{\beta E\}$

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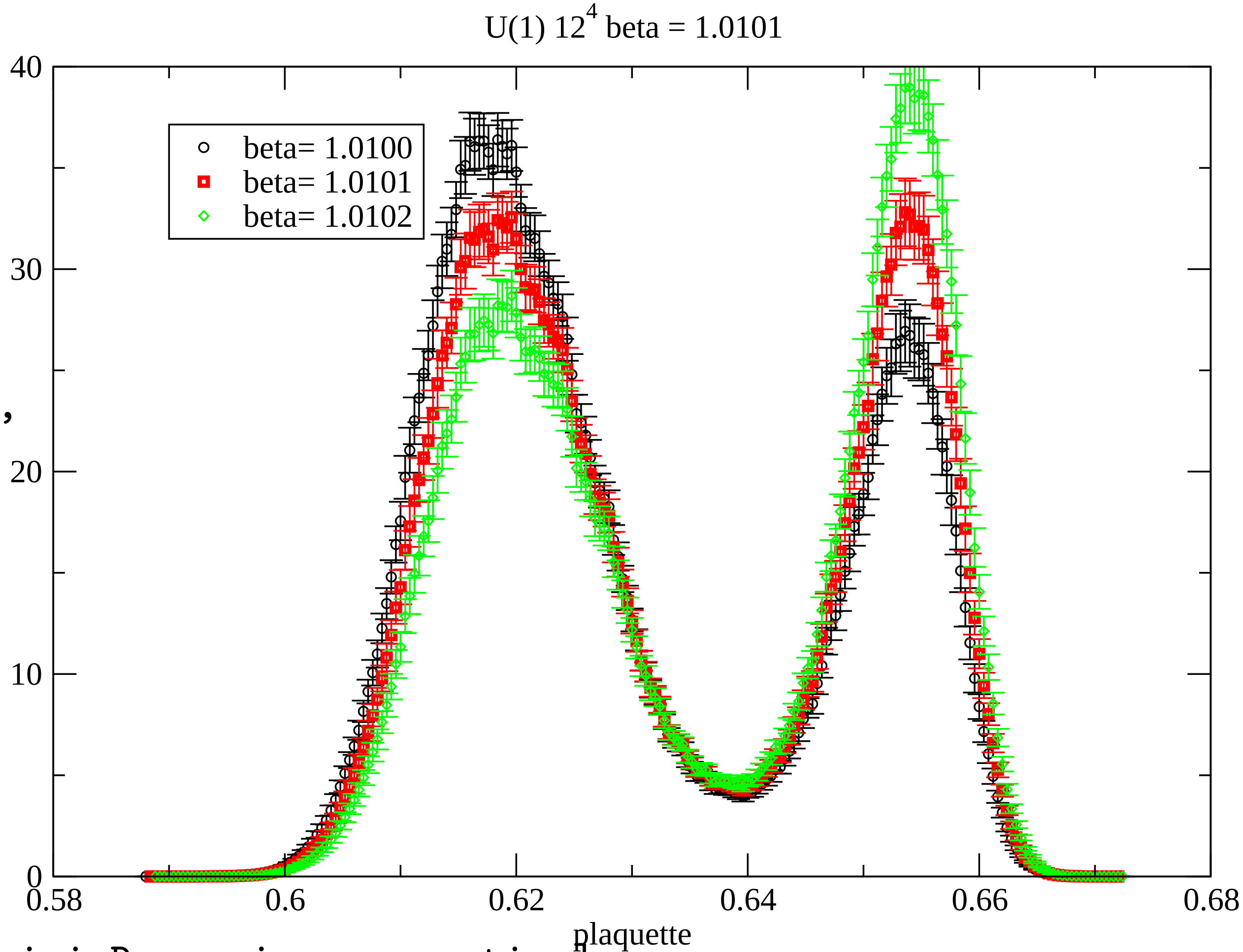

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`[ Arnold, Bunk, Lippert, Schilling, Nucl.Proc.Proc.Suppl. 119 (2003) 119]`

△ $P_\beta(E) = \rho(E)\exp\{\beta E\}$

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Solving sign-problems

Can we use the LLR method to calculate the Polyakov loop effective potential?

The generalised LLR approach

- The generalised density-of-states approach:

$$\rho(q) \;=\; \int \mathcal{D}U_\mu \;\exp\{S_{\text{pure}}[U]\}\; \delta\Big(q - p[U]\Big)$$

number of states for a given Polyakov line

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- Recovering the generating functional:

$$Z[j] \;=\; \int dq \; \rho(q) \;\; \exp\{j\, q\,\}.$$

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- Recovering the generating functional:

$$Z[j] \;=\; \int dq \;\rho(q)\; \exp\{j\, q\,\}.$$

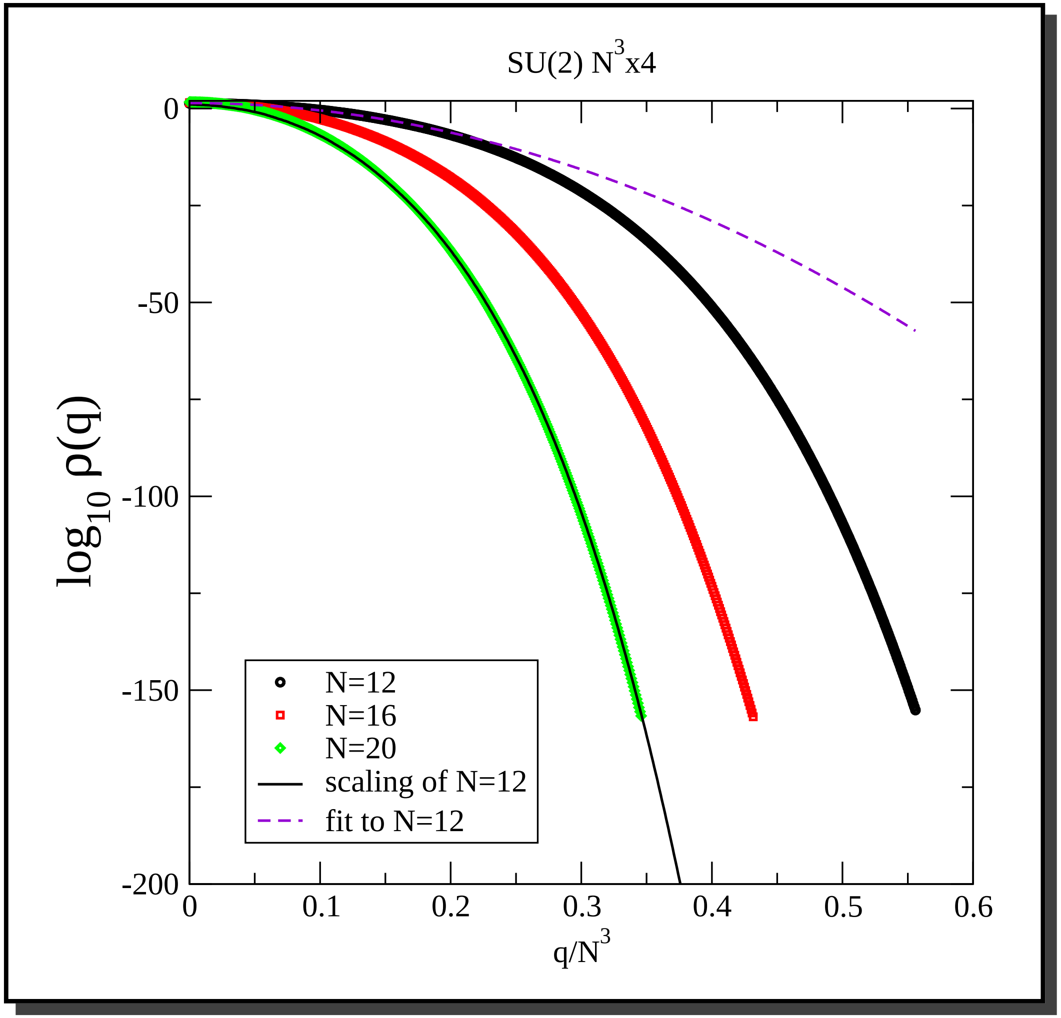
- Use a piecewise linear ansatz:

$$\rho(q) \;=\; \rho(q_0)\; \exp\Big[a(q_0)(q-q_0)\Big]$$

calculate $a(q_0)$ with the LLR method

Results - Polyakov line effective potential

- Polyakov line probability distribution:

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[Langfeld, Pawlowski,
Two-colour QCD with heavy
quarks at finite densities,
arXiv:1307.0455 [hep-lat]]
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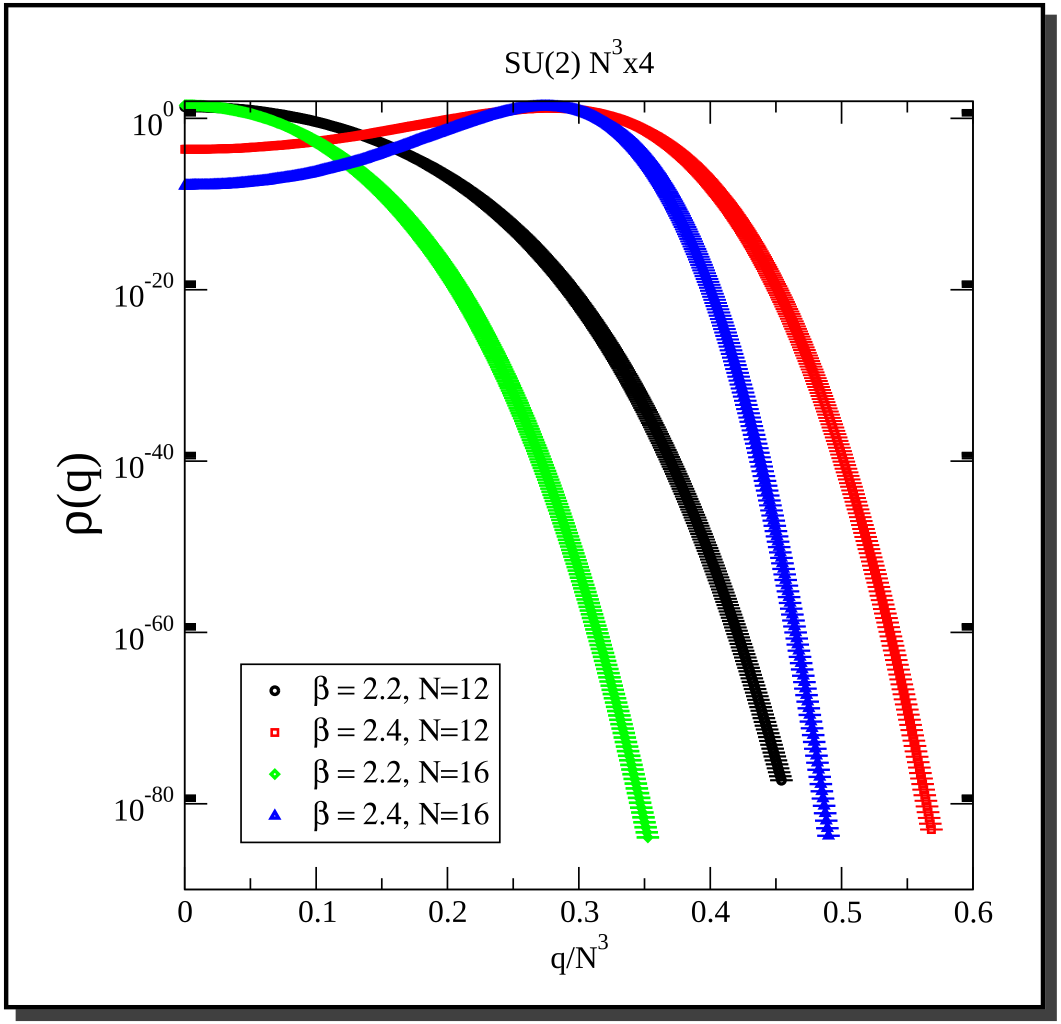
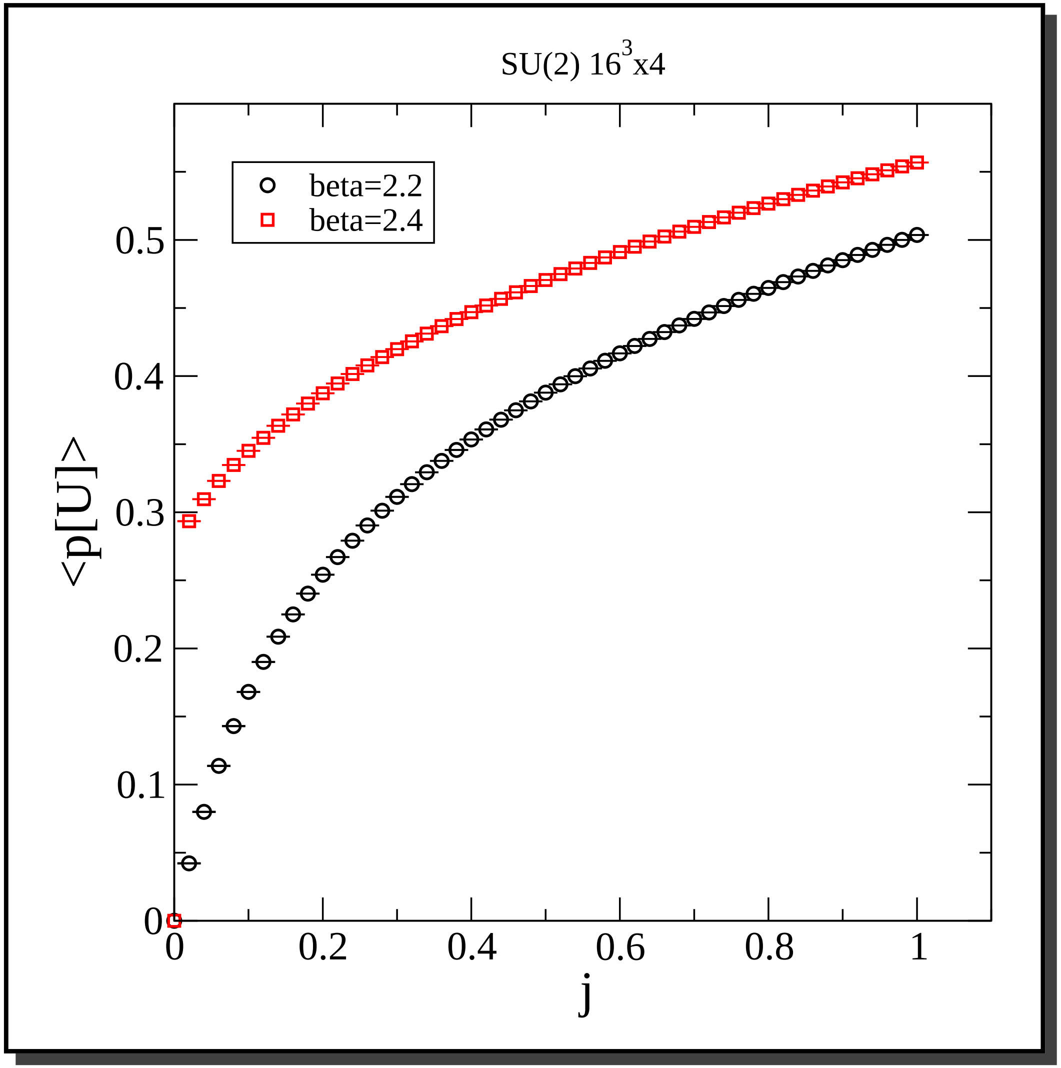
Results - Polyakov line effective potential

- Polyakov line probability distribution - finite temperatures



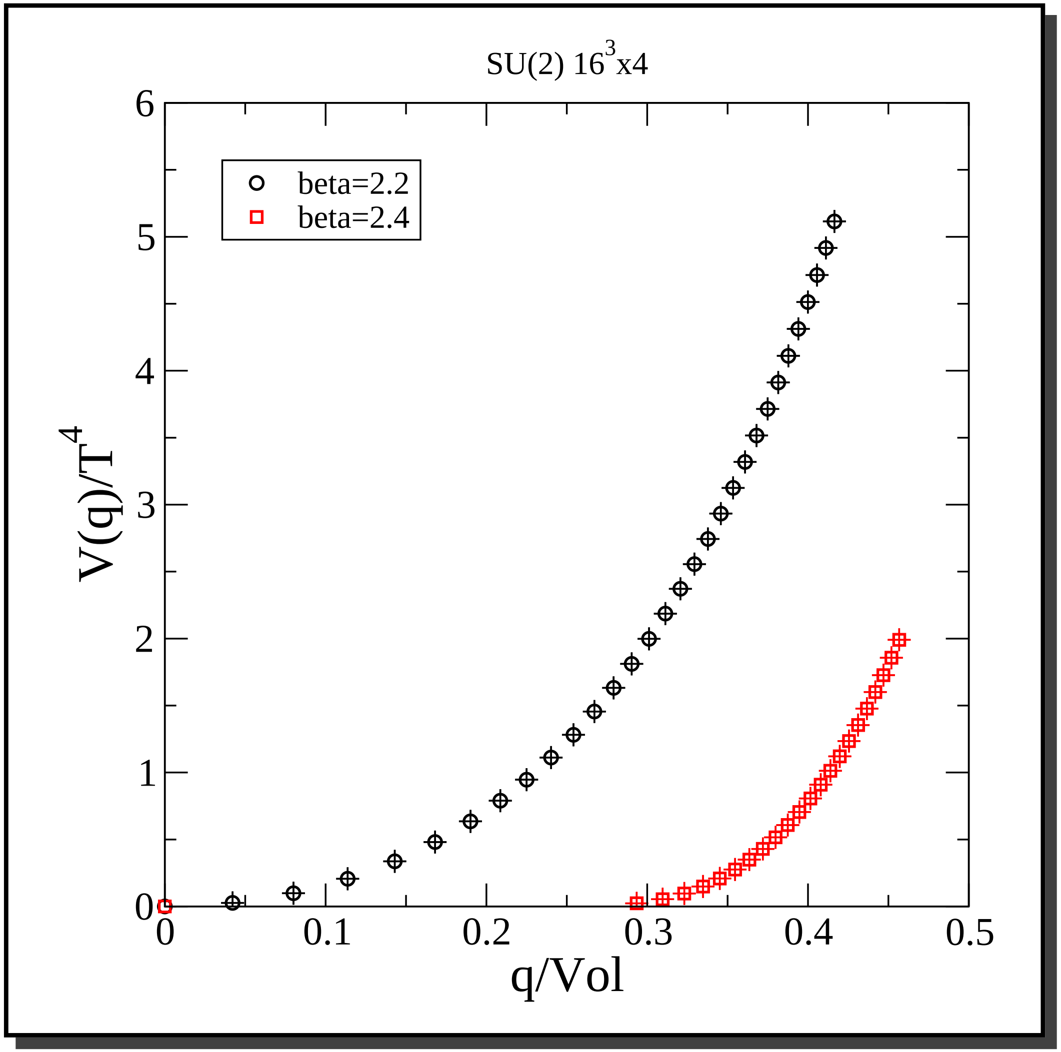
Results - Polyakov line effective potential

- Polyakov line expectation value - zero and finite T



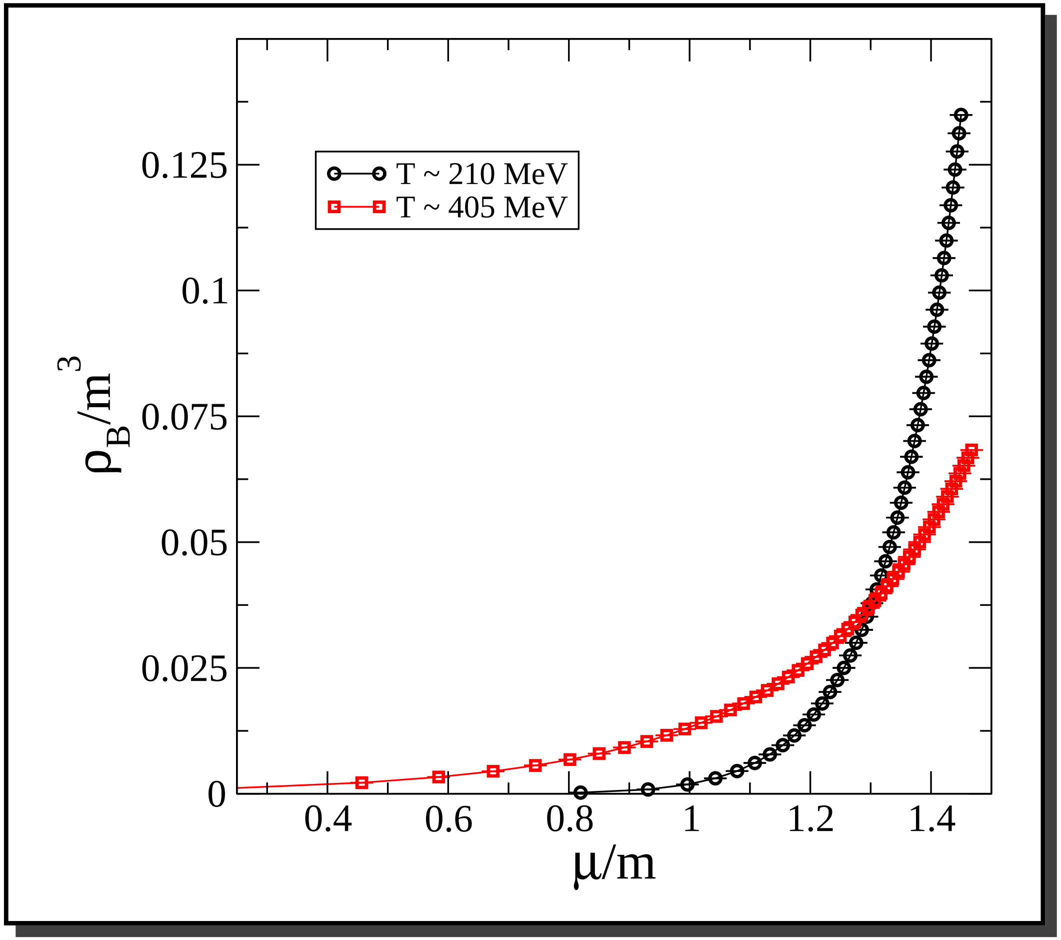
Results - Polyakov line effective potential

- Coleman effective potential - zero and finite temperatures



Results - Polyakov line effective potential

- SU(2) “charmonium” - zero and finite temperatures



Conclusions

- studied SU(2) YM-theory at finite densities for heavy quarks systematic $1/m$ expansion $\Rightarrow$ "weak coupling" version of the Polyakov line spin model

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  need to solve an overlap problem
  `[poor signal-to-noise ratio due to large cancellation]`

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  systematic $1/m$ expansion $\Rightarrow$
  "weak coupling" version of the Polyakov line spin model
- Coleman effective potential:
  need to solve an overlap problem
  `[poor signal-to-noise ratio due to large cancellation]`
- We might have a "first priciples" method to solve
  overlap problems!
  `[Langfeld, Lucini, Rago, PRL 109 (2012) 111601]`

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- Here:

  △ Polyakov loop probability distribution over 80 orders of magnitude due to exponential error suppression inherent to LLR method

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- Outlook: LLR approach not restricted to real actions!

  will study of the O(2) model at finite densities (has a dual theory that is real!)

  `[Langfeld, Phase diagram of the quantum O(2)-model in 2+1 dimensions, PRD 87, 114504 (2013)]`