Banks-Casher-type relations for complex Dirac spectra

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Outline

- Banks-Casher relation for QCD (1980):

$$|\langle\bar{\psi}\psi\rangle| = \pi\rho(0)$$

  $\rho(\lambda) =$ spectral density of Dirac operator (λ real)

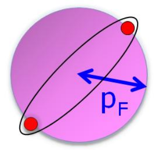
- links spontaneous breaking of chiral symmetry to accumulation of Dirac eigenvalues near zero ($\lambda_{\text{min}} \sim 1/V$)
- useful for lattice, e.g., talk by Georg Engel (Parallels 7D)
- if fermionic measure is not positive definite (e.g., chemical potential $\mu \neq 0$)
  - derivation breaks down, $\rho(0)$ undefined — Leutwyler-Smilga 1992
  - connection between $\langle\bar{\psi}\psi\rangle$ and (complex) Dirac spectrum more complicated — Osborn-Splittorff-Verbaarschot 2005-2008
- this talk:
  - consider QCD-like theories with complex Dirac spectrum but positive definite fermionic measure
  - derive BC-type relations involving $\rho(\lambda = 0)$ with $\lambda \in \mathbb{C}$
  - high density, zero temperature

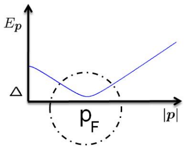
- perturbative calculations at large μ: Son 1999, T. Schäfer 2000

$$0 \approx \langle \bar{\psi}\psi \rangle \ll \langle \psi\psi \rangle$$

$\rightarrow$ chiral symmetry and $\mathrm{U}(1)_B$ broken by the diquark condensate

- for $\mu \gg \Lambda_{\mathrm{QCD}}$ we have BCS-type diquark pairing (since there is an attractive channel between quarks near the Fermi surface)
- diquarks are loosely bound in real space





- diquark pairing pattern depends on theory (QCD, QCD_2, isospin, adjoint)

- we consider QCD-like theories with complex Dirac spectrum, but without sign problem, in three symmetry classes:
  - two-color QCD at high quark density ($\beta_{\text{Dyson}} = 1$)
  - QCD at high isospin density ($\beta_{\text{Dyson}} = 2$)
  - adjoint QCD at high quark density ($\beta_{\text{Dyson}} = 4$)

BC-type relation:

$$\Delta^2 = \frac{2\pi^3}{3d_{\text{rep}}}\rho(0)$$

- $\Delta =$ BCS gap
- $\rho(\lambda) =$ two-dimensional spectral density
- $d_{\text{rep}} =$ dimension of color representation ($d_{\text{fund}} = N_c$, $d_{\text{adj}} = N_c^2 - 1$)

- main idea:
  - write down partition function $Z(M)$ as a function of quark masses in fundamental (QCD-like) theory and in low-energy effective theory
  - take suitable derivatives w.r.t. quark masses in both theories (yields $\sim \rho(0)$ in one case and $\sim \Delta^2$ in the other case)
  - identify the results
- for the derivation, low-energy effective theory is needed
  - previously known for QCD$_2$, constructed here for isospin and adjoint
  - explicit breaking of U$(1)_A$ is strongly suppressed at high density instead, U$(1)_A$ is assumed to be spontaneously broken by $\langle\psi\psi\rangle \neq 0$
  - QCD inequalities allow us to exclude certain symmetry-breaking patterns (only applicable if measure is positive definite)
  - coefficient of mass term is not a free parameter but can be computed in high-density effective theory (HDET: D.K. Hong 1998-2000)

Outline

- Dirac operator:

$$D(\mu) = \gamma_\nu D_\nu + \mu\gamma_4$$

- partition function:

$$Z = \Big\langle \prod_{f=1}^{N_f} (D + m_f) \Big\rangle_{\text{YM}}$$

$$\underset{\substack{\uparrow \\ N_f = 2}}{=} \big\langle \det(D + m_1)\det(D + m_2) \big\rangle_{\text{YM}} \underset{\substack{\uparrow \\ [C\gamma_5\tau_2 K, D] = 0}}{=} \big\langle \det(D + m_1)\det(D^\dagger + m_2) \big\rangle_{\text{YM}}$$

- two possibilities to have positive definite measure:
  - $m_1 = m_2 = m \in \mathbb{R}$ (leads to mathematical subtleties in derivation of BC)
  - $m_1 = z$, $m_2 = z^*$ with $z \in \mathbb{C}$ (allows for rigorous derivation of BC)

$\longrightarrow$ choose second possibility

- partition function in terms of Dirac eigenvalues (with regulator ε^2):

$$Z = \left\langle \prod_n \left[(\lambda_n + z)(\lambda_n^* + z^*) + \varepsilon^2 \right] \right\rangle_{\text{YM}}$$

- defining the spectral density and connected two-point function

$$\rho(\lambda) = \lim_{z\to 0} \lim_{V_4\to\infty} \frac{1}{V_4} \left\langle \sum_n \delta^2(\lambda - \lambda_n) \right\rangle_{N_f=2}$$

$$\rho_2^c(\lambda,\lambda') = \lim_{z\to 0} \lim_{V_4\to\infty} \left[\frac{1}{V_4} \left\langle \sum_m \delta^2(\lambda-\lambda_m) \sum_n \delta^2(\lambda'-\lambda_n) \right\rangle_{N_f=2} - V_4 \rho(\lambda)\rho(\lambda') \right]$$

we obtain

$$\lim_{z\to 0} \lim_{V_4\to\infty} \frac{1}{V_4} \frac{\partial^2 \log Z}{\partial z \partial z^*} = \int d^2\lambda \, \frac{\varepsilon^2}{(|\lambda|^2+\varepsilon^2)^2} \rho(\lambda) + \int d^2\lambda \int d^2\lambda' \, \frac{\lambda^*}{|\lambda|^2+\varepsilon^2} \frac{\lambda'}{|\lambda'|^2+\varepsilon^2} \rho_2^c(\lambda,\lambda')$$

- the last line vanishes due to chiral symmetry

$$\rho_2^c(\lambda,\lambda') = \rho_2^c(-\lambda,\lambda') = \rho_2^c(\lambda,-\lambda') = \rho_2^c(-\lambda,-\lambda')$$

- using the delta function in the complex plane

$$\delta^2(z) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{(|z|^2+\varepsilon^2)^2}$$

we end up with

$$\lim_{\varepsilon\to 0}\lim_{z\to 0}\lim_{V_4\to\infty}\frac{1}{V_4}\,\partial_{z^*}\partial_z \log Z = \pi\rho(0)$$

- was constructed in Kanazawa-Wettig-Yamamoto 0906.3579
  here: small technical modifications to allow for complex $M = \mathrm{diag}(z, z^*)$

- shift in free energy due to BCS pairing of quarks near the Fermi surface:

$$\delta\mathscr{E} = \min_{A \in \mathrm{U}(1)} \left\{ -\frac{3}{2\pi^2} \Delta^2 (A^2 + A^{*2}) \det M \right\}$$

  for $M = \mathrm{diag}(z, z^*)$ the minimum is obtained for $A = \pm 1$

- together with a high-energy term $H_2 \operatorname{tr} M^2$ we obtain

$$\frac{1}{V_4} \log Z = \frac{3}{\pi^2} \Delta^2 z z^* + H_2 (z^2 + z^{*2}) + O(|z|^3)$$

  and thus

$$\lim_{z \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \partial_{z^*} \partial_z \log Z = \frac{3}{\pi^2} \Delta^2 \underset{\substack{\uparrow \\ \text{fund. theory}}}{=} \pi \rho(0) \qquad \square$$

Some technical comments

- contribution of zero modes is subleading in $1/V_4$ (just as at $\mu = 0$)
  (in addition, topology is strongly suppressed at large μ)
- regulator ε was necessary to avoid singularities in $1/(\lambda_n + z)$
  (at $\mu = 0$, terms like $1/(i\lambda_n + m)$ are never singular)
- there is a UV divergence in the integral over $d^2\lambda$
  - must be regularized and drops out if $\varepsilon \to 0$ is taken before $\Lambda_{\text{UV}} \to \infty$
- real masses m_1 and m_2:
  - with $m_1 = m_2 = m$ the Δ^2 and H_2 terms would mix
    $\to$ cannot extract Δ^2 by derivatives w.r.t. m
  - with $\lim_{m_{1,2}=m} \partial_{m_1} \partial_{m_2} \log Z$ the fermionic measure would not have been positive definite at all stages of the calculation
- at high density $g \ll 1$, and H_2 can be computed to leading order in g:

$$H_2 = -\frac{d_{\text{rep}}}{4\pi^2}(\mu^2 + \Lambda_{\text{UV}}^2)$$

(last term is scheme-dependent)

General even N_f

- mass matrix is now $M = \mathrm{diag}(z,\dots,z,z^*,\dots,z^*)$ with $N_f/2$ entries each of z and z^*
- on the low-energy effective theory side we now have 0906.3579

$$\delta\mathcal{E} = -\frac{3\Delta^2}{4\pi^2} \max_{A,\Sigma_L,\Sigma_R} \mathrm{Re}\left\{A^2\,\mathrm{tr}(M\Sigma_R M^T\Sigma_L^\dagger) + A^{*2}\,\mathrm{tr}(M\Sigma_L M^T\Sigma_R^\dagger)\right\}$$

  with $A \in \mathrm{U}(1)$ and $\Sigma_{L,R} \in \mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$
- additional terms are allowed by symmetries, but their coefficients are zero in HDET
- the maximum is obtained for $A = \pm 1$ and (by using a Cauchy-Schwarz inequality for matrices)

$$\Sigma_{L,R} = \pm\begin{pmatrix} 0 & -\mathbb{1}_{N_f/2} \\ \mathbb{1}_{N_f/2} & 0 \end{pmatrix}$$

- we obtain

$$\delta\mathscr{E} = -\frac{3N_f}{2\pi^2}\Delta^2 z z^*$$

and thus

$$\lim_{z\to 0}\lim_{V_4\to\infty}\frac{1}{V_4}\partial_{z^*}\partial_z \log Z = \frac{3N_f}{2\pi^2}\Delta^2$$

- on the fundamental theory side, derivation goes through in the same way as for $N_f = 2$ and yields

$$\lim_{\varepsilon\to 0}\lim_{z\to 0}\lim_{V_4\to\infty}\frac{1}{V_4}\partial_{z^*}\partial_z \log Z = \frac{N_f}{2}\pi\rho(0)$$

- this results in the same BC-type relation as for $N_f = 2$

Outline

- partition function of QCD with $N_c \geq 2$ colors and $N_f = 2$ flavors with isospin chemical potential $\mu_I = 2\mu$:

$$\begin{aligned} Z &= \left\langle \det(D(\mu) + m_1) \det(D(-\mu) + m_2) \right\rangle_{\text{YM}} \\ &= \left\langle \det(D + m_1) \det(D^\dagger + m_2) \right\rangle_{\text{YM}} \end{aligned}$$

since $D(-\mu) = -D(\mu)^\dagger$ and the Dirac eigenvalues occur in pairs $\pm\lambda$

- in the fundamental theory and with $M = \text{diag}(z, z^*)$, we again obtain

$$\lim_{\varepsilon \to 0} \lim_{z \to 0} \lim_{V_4 \to \infty} \frac{1}{V_4} \partial_{z^*} \partial_z \log Z = \pi\rho(0)$$

- the low-energy effective theory at high density was not known before

- at large μ_I, BCS pairing of type $\langle \bar{d}\gamma_5 u\rangle$ occurs near Fermi surface
  Son-Stephanov 2001
- analysis similar to QCD_2, but coset space is now $\mathrm{U}(1)_A \times \mathrm{U}(1)_{I_3}$
- free-energy shift is now, with $A \in \mathrm{U}(1)$,

$$\delta\mathscr{E} = -c_{\text{iso}}\Delta^2 \max_A \left\{(A^2 + A^{*2}) \det M\right\} \underset{\substack{\uparrow \\ A=\pm 1}}{=} -2c_{\text{iso}}\Delta^2 zz^*$$

- c_{iso} can be computed in HDET, similar to T. Schäfer 2001 and KWY 2009
  after some calculations (using Hanada-Yamamoto 2011) we obtain

$$c_{\text{iso}} = \frac{3N_c}{4\pi^2}$$

- this gives the BC-type relation

$$\Delta^2 = \frac{2\pi^3}{3N_c}\rho(0)$$

Outline

1. Introduction
2. Two-color QCD
   - Derivation of BC-type relation for $N_f = 2$
   - Generalization to general (even) N_f
3. QCD at nonzero isospin density
4. Adjoint QCD
5. Consistency with microscopic limit and RMT
6. Summary

- fermions now transform in adjoint representation of $\mathrm{SU}(N_c)$ with $N_c \geq 2$
- mass matrix is $M = \mathrm{diag}(z,\dots,z,z^*,\dots,z^*)$ again, with N_f even
- partition function:

$$\begin{aligned} Z &= \left\langle \det{}^{N_f/2}(D+z)\det{}^{N_f/2}(D+z^*)\right\rangle_{\mathrm{YM}} \\ &\underset{\substack{\uparrow \\ [C\gamma_5 K, D]=0}}{=} \left\langle \det{}^{N_f/2}(D+z)\det{}^{N_f/2}(D^\dagger+z^*)\right\rangle_{\mathrm{YM}} \end{aligned}$$

- in the fundamental theory, we again obtain

$$\lim_{\varepsilon\to 0}\lim_{z\to 0}\lim_{V_4\to\infty}\frac{1}{V_4}\partial_{z^*}\partial_z \log Z = \frac{N_f}{2}\pi\rho(0)$$

- we still need to construct the low-energy effective theory at high density

- condensation occurs in pseudoscalar channel Kanazawa, PhD 2011
  diquark condensate symmetric in color and flavor, antisymmetric in spin
- coset space is now KWY 1110.5858
  $\mathrm{U}(1)_B \times \mathrm{U}(1)_A \times [\mathrm{SU}(N_f)_L/\mathrm{SO}(N_f)_L] \times [\mathrm{SU}(N_f)_R/\mathrm{SO}(N_f)_R]$
- free-energy shift, with $A \in \mathrm{U}(1)$ and $\Sigma_{L,R} \in \mathrm{SU}(N_f)/\mathrm{SO}(N_f)$

$$\delta\mathscr{E} = -c_{\text{adj}}\Delta^2 \max_{A,\Sigma_L,\Sigma_R} \mathrm{Re}\left\{A^2\,\mathrm{tr}(M\Sigma_R M^T\Sigma_L^\dagger) + A^{*2}\,\mathrm{tr}(M\Sigma_L M^T \Sigma_R^\dagger)\right\}$$

$$= -c_{\text{adj}}\Delta^2 \cdot 2N_f z z^* \qquad \text{for} \quad A = \pm 1\,, \quad \Sigma_{L,R} = \pm\begin{pmatrix} 0 & \mathbb{1}_{N_f/2} \\ \mathbb{1}_{N_f/2} & 0 \end{pmatrix}$$

- additional terms are allowed by symmetries, but coefficients $= 0$ in HDET
- computation of c_{adj} in HDET yields $c_{\text{adj}} = \dfrac{3(N_c^2-1)}{8\pi^2}$
- this gives the BC-type relation

$$\Delta^2 = \frac{2\pi^3}{3(N_c^2-1)}\rho(0)$$

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6. Summary

- microscopic density ρ_s = spectral density magnified to resolve individual eigenvalues
  - ρ_s is universal and can be computed in RMT
- to match physical theory and RMT, microscopic scale has to be set by properly rescaling the Dirac eigenvalues
  - this can be done, e.g., by matching the mass-dependence of the corresponding partition functions
- in all three cases the rescaling is given by

$$\rho_s(\xi) = \lim_{V_4\to\infty} \frac{2\pi^2}{3d_{\text{rep}}\Delta^2}\,\rho\left(\sqrt{\frac{2\pi^2}{3d_{\text{rep}}V_4\Delta^2}}\;\xi\right)$$

- RMT results for all three cases follow from known results in the literature
  - we have $\lim\limits_{|\xi|\to\infty}\rho_s(\xi) \to \dfrac{1}{\pi}$, which is consistent with $\Delta^2 = \dfrac{2\pi^3}{3d_{\text{rep}}}\rho(0)$

- derived BC-type relation for QCD-like theories without sign problem at high density (QCD_2, isospin, adjoint):

$$\Delta^2 = \frac{2\pi^3}{3d_{\text{rep}}}\rho(0)$$

- constructed the corresponding low-energy effective theories
  - coefficient of mass term is not a free parameter but can be computed in HDET
- result is consistent with microscopic limit and RMT
- BC-type relation useful to determine Δ on the lattice (together with spectral sum rules and microscopic spectral correlations)
- possible extensions:
  - low density $\rightarrow$ $\rho(0)$ related to chiral susceptibility and (combination of) LECs
  - nonzero temperature $\rightarrow$ solve gap equation and compute mass dependence of free energy in terms of Δ at $T \neq 0$
  - CFL phase of QCD $\rightarrow$ similar singular behavior of ρ near origin as at low μ?