Adler function and hadronic vacuum polarization from lattice vector correlators in the time-momentum representation

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Outline

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  - Recent numerical results for $\Pi(Q^2)$ on F6 ensemble
  - Newest numerical results for a_μ^{HLO} in $N_f = 2$
- Hadronic vacuum polarization based on the time-momentum representation of the vector correlator
  - The lattice isovector vector correlator
  - Numerical results for $\hat{\Pi}(Q^2)$ and $d\hat{\Pi}(Q^2)/dQ^2$
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  - Effect in the electromagnetic spectral function
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- Overview

Hadronic vacuum polarization from the 4D Fouriertransform of the vector correlator

- On a Euclidean lattice the vacuum polarization tensor can be defined as the four dimensional Fouriertransform of the vector current-current correlation function

$$\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x} \langle j_\mu(x) j_\nu(0)\rangle$$

- The tensor structure implies

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right)\Pi(Q^2)$$

- The vacuum polarisation $\Pi(Q^2)$ can be computed from the lattice determined $\Pi_{\mu\nu}(Q)$
- Caveat: Only a limited number of $Q^2_{latt.}$ is available
  $\Rightarrow$ Can be boosted by "twisted boundary conditions"

Anomalous magnetic moment of the muon a_μ^{HLO}

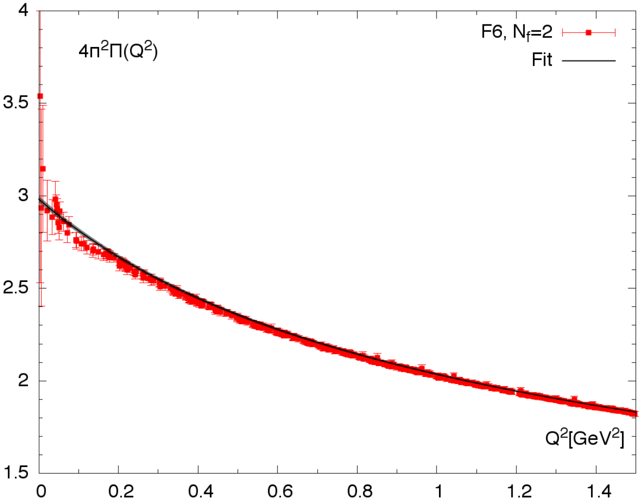
- Lowest order hadronic contribution to the anomalous magnetic moment of the muon a_μ^{HLO} is obtained by integrating

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dQ^2 K_E(Q^2, m_\mu)\hat{\Pi}(Q^2)$$

- here, the hadronic part $\hat{\Pi}(Q^2) = 4\pi^2(\Pi(Q^2) - \Pi(0))$ can be determined by taking the limit $\lim_{Q^2\to 0}\Pi(Q^2)$
- and the kernel given by QED is

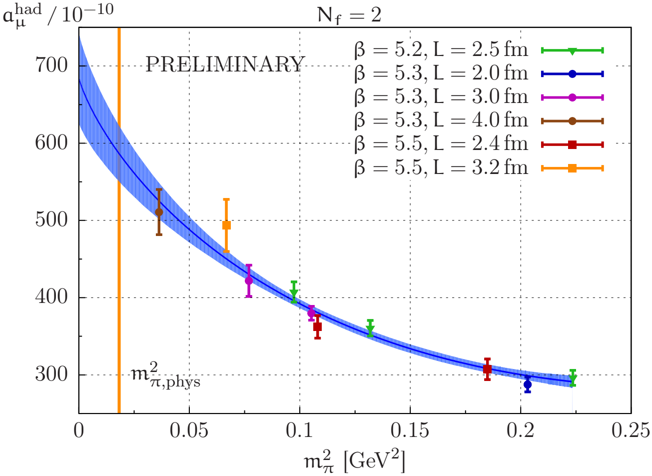
$$K_E(Q^2, m_\mu) = \frac{m_\mu^2 Q^2 Z^3(1 - Q^2 Z)}{1 + m_\mu^2 Q^2 Z^2}, \quad Z = \frac{Q^2 - \sqrt{Q^4 - 4m_\mu^2 Q^2}}{2m_\mu^2 Q^2}$$

Example: Recent numerical results for $\Pi(Q^2)$ on F6



- For $\hat{\Pi}(Q^2)$ the point $\Pi(0)$ has to be obtained from a fit

Newest numerical results for a_μ^{HLO} in $N_f = 2$



- a_μ^{HLO} can be determined and chirally extrapolated
  - χPT inspired fit: $A + B\, m_\pi^2 + C\, m_\pi^2\, \ln(m_\pi^2)$

Recap of the "standard method"

- $\Pi(Q^2)$ is extracted from the 4D Fourier transformation of the vector correlation function

$$\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x}\langle j_\mu(x) j_\nu(0)\rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi(Q^2)$$

- The limit $\lim_{Q^2\to 0}\Pi(Q^2)$ is estimated to obtain

$$\hat{\Pi}(Q^2) = 4\pi^2(\Pi(Q^2) - \Pi(0))$$

- $\hat{\Pi}(Q^2)$ is used to determine a_μ^{HLO}, through

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dQ^2 K_E(Q^2, m_\mu)\hat{\Pi}(Q^2)$$

Obtaining $\hat{\Pi}(Q_0^2)$ from the time-momentum vector correlator

- The tructure of the vacuum polarization tensor implies:

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right)\Pi(Q^2)$$
$$\Rightarrow \Pi_{zz}(Q_0) = -Q_0^2 \Pi(Q_0^2) \quad \text{whereby: } Q_0 = Q(\omega, \vec{k} = 0)$$

- The time-momentum representation of the vector correlation function is given by:

$$G(x_0, \vec{k}) = \int_x d^3x\, e^{i\vec{k}\vec{x}} \langle J_\mu(x_0, \vec{x}) J_\nu(0) \rangle$$

- Therefore $\Pi(Q_0^2)$ can be rewritten in terms of the mixed-representation correlator as:

$$\Pi(Q_0^2) = -\frac{\Pi_{zz}(Q_0)}{Q_0^2} = \frac{1}{Q_0^2} \int_{-\infty}^{\infty} dx_0\, e^{iQ_0 x_0} G(x_0, \vec{k} = 0)$$

Obtaining $\hat{\Pi}(Q_0^2)$ from the time-momentum vector correlator

- To obtain $\hat{\Pi}(Q_0^2) = 4\pi^2(\Pi(Q_0^2) - \Pi(0))$ expand:

$$\Pi(Q_0^2 \to 0) = \frac{1}{Q_0^2}\int_{-\infty}^{\infty} dx_0\, G(x_0) \\ - \frac{1}{2}\int_{-\infty}^{\infty} dx_0\, x_0^2\, G(x_0) + \mathcal{O}(Q_0^2)\ldots$$

- The vacuum polarization can be expressed as an integral over the current-current correlator $G(x_0)$:

$$\Pi(Q_0^2) - \Pi(0) = \int_0^{\infty} dx_0\, G(x_0)\left[x_0^2 - \frac{4}{Q_0^2}\sin^2\left(\frac{1}{2}Q_0 x_0\right)\right]$$

Vacuum polarization from the mixed representation vector correlator

- There is no extrapolation to $Q_0^2 = 0$ needed to obtain $\hat{\Pi}(Q_0^2)$
- There is no limitation to a finite number of lattice momenta
- Also the Adler function can be computed directly:

$$D(Q_0^2) \equiv 12\pi^2 Q_0^2 \frac{d\,\Pi}{dQ_0^2} = \frac{12\pi^2}{Q_0^2} \int_0^\infty dx_0\, G(x_0) \\ (2 - 2\cos(Q_0 x_0) - Q_0 x_0 \sin(Q_0 x_0))$$

- The slope of the Adler function at the origin is

$$D'(0) = \lim_{Q^2 \to 0} \frac{D(Q^2)}{Q^2} = \pi^2 \int_0^\infty dx_0\, x_0^4\, G(x_0)$$

$$\Rightarrow \lim_{m_l \to 0} \frac{a_l^{HLO}}{m_l} = \frac{1}{9}\left(\frac{\alpha}{\pi}\right)^2 D'(0)$$

Numerical Setup

- F6 ensemble: 96×48^3, $a = 0.0631\text{fm}$ at $m_\pi = 324\text{MeV}$
- Local-conserved isovector vector correlation function:

$$G(x_0) = Z_V(g_0) G^{\mathrm{bare}}(x_0, g_0)\delta_{kl} = -a^3 Z_V(g_0) \sum_{\vec{x}} \langle J_k^c(x) J_\ell^l(0)\rangle,$$

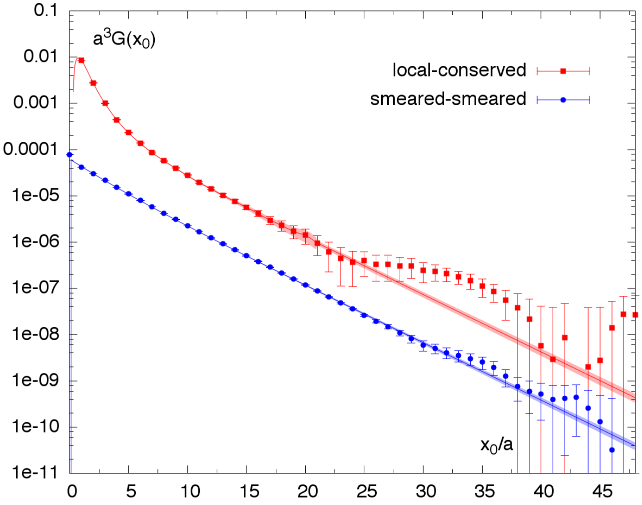
where

$$\begin{aligned} J_\mu^l(x) &= \bar{q}(x)\gamma_\mu q(x) \\ J_\mu^c(x) &= \frac{1}{2}\Big(\bar{q}(x + a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)q(x) \\ &\quad - \bar{q}(x)(1-\gamma_\mu)U_\mu(x)q(x+a\hat{\mu})\Big) \end{aligned}$$

- To extrapolate to $x_0 \to \infty$ use Ansatz:

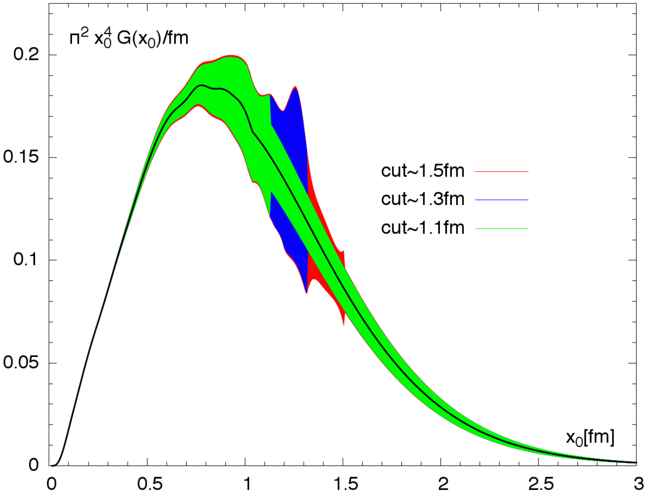
$$G_{\mathrm{Ansatz}}(x_0) = \sum_{n=1}^{2} |A_n|^2 e^{-m_n x_0}$$

Lattice isovector vector correlation functions



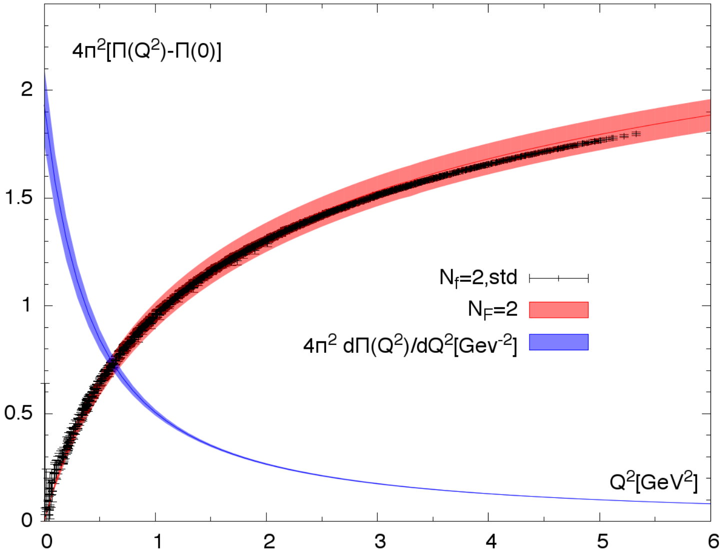
- The smeared-smeared correlator is used to fit the lowest lying mass for extrapolation to all time beyond $x_0 \simeq T/4$.

Integrand for computing the slope $d\hat{\Pi}(Q^2)/dQ^2$

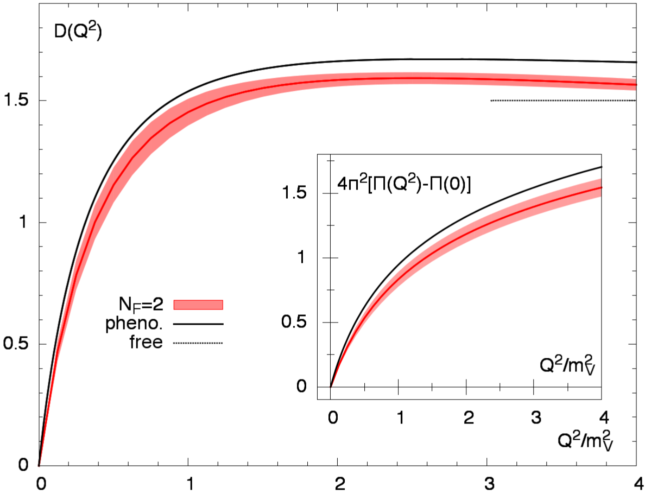


- Different cuts lead to negligible effects in the result on $d\hat{\Pi}(Q^2)/dQ^2$ and also $\hat{\Pi}(Q^2)$.

Lattice results: $\hat{\Pi}(Q^2)$ and $d\hat{\Pi}(Q^2)/dQ^2$

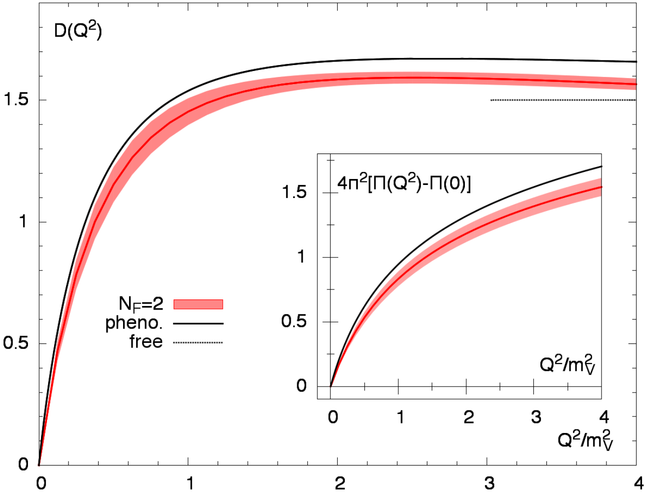


Lattice results: $\hat{\Pi}(Q^2)$ and Adler function



- Note: Horizontal axis rescaled by the vector meson mass

Lattice results: $\hat{\Pi}(Q^2)$ and Adler function



- Lattice results low compared to phenomenological model[†]
  $\Rightarrow$ Due to spectral density below and around ρ mass?

[†]: Eq. (93) of Eur.Phys.J. A47, 148(2011).

On the role of disconnected diagrams

- In (isospin-symmetric) two-flavor lattice QCD correlation functions can written in terms of Wick-connected and Wick-disconnected diagrams
- So far we have concentrated on the isovector channel:

$$\Pi^{\rho\rho}_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x} \langle j_\mu(x) j_\nu(0)\rangle = \frac{1}{2}\Pi^{wick-conn.}_{\mu\nu}(Q)$$

$\Rightarrow$ it contains only connected diagrams

- The electromagnetic current however is

$$\Pi^{\gamma\gamma}_{\mu\nu}(Q) = \Pi^{\rho\rho}_{\mu\nu}(Q) + \frac{1}{9}\Pi^{\omega\omega}_{\mu\nu}(Q)$$

where: $\Pi^{\omega\omega}_{\mu\nu}(Q) = \frac{1}{2}\Pi^{Wick-conn.}_{\mu\nu}(Q) + \Pi^{Wick-disconn.}_{\mu\nu}(Q)$

On the role of disconnected diagrams

- Both $\Pi_{\mu\nu}^{wick-conn.}(Q)$ and $\Pi_{\mu\nu}^{wick-disconn.}(Q)$ can be linked to a spectral function via $\pi\rho(s) = -\mathrm{Im}\Pi(Q^2)$
- The spectral function $\rho(s)$ is related to the experimentally accessible $R(s) \sim \sigma(e^+e^- \to \mu^+\mu^-)$ ratio, some implications are:
  - Isovector: $\rho^{\rho\rho}(\sqrt{s} < 2m_\pi) = 0$
  - Isosinglett: $\rho^{\omega\omega}(\sqrt{s} < 3m_\pi) = 0$
- In terms of Wick-contractions this implies for $\sqrt{s} < 3m_\pi$

$$\rho^{Wick-disconn.}(s) = -\frac{1}{2}\rho^{Wick-conn.}(s)$$

- Therefore in the electromagnetic current at $2m_\pi < \sqrt{s} < 3m_\pi$

$$\frac{\frac{1}{9}\rho^{Wick-disconn.}(s)}{\frac{5}{9}\rho^{Wick-conn.}(s)} = -\frac{1}{10}$$

On the role of disconnected diagrams

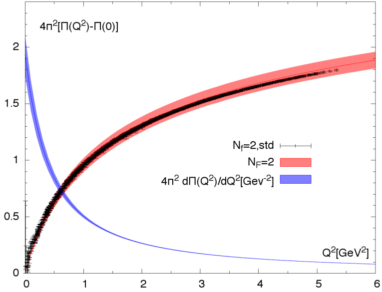
- The result $\rho^{Wick-disconn.}(s) = -\frac{1}{2}\rho^{Wick-conn.}(s)$ can be translated into a Euclidean correlator using:

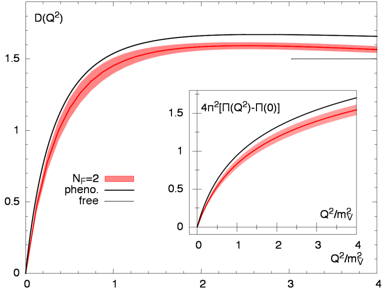
$$G(x_0) = \int_0^\infty d\sqrt{s} \;\; s \; \rho(s) \; e^{-\sqrt{s}|x_0|}$$

- The correlator at long distances is dominated by the low-energy part of the spf. For $x_0 \to \infty$ it follows:

$$G^{Wick-disconn.}(x_0) = -\frac{1}{2} G^{Wick-conn.}(x_0)\Big[1 + \mathcal{O}(e^{-m_\pi x_0})\Big]$$

Overview





- We implemented a new representation of the hadronic vacuum polarization, that
  - does not require an extrapolation $Q^2 \to 0$
  - is not limited to a finite number of lattice momenta
  - enables the direct computation also of the Adler function and the slope of $\hat{\Pi}(Q^2)$
- We independently rederived a recent theoretical estimate of the Wick-disconnected diagram contributions