Computing the Adler function from the vacuum polarization

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Outline

1. Adler function
   - Methods to compute the Adler function
   - Numerical results for the Adler function
2. Ward identity
   - Partially twisted boundary conditions
   - Vacuum polarization
   - Numerical results
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Adler function

The Adler function is defined as[1]

$$\frac{D(q^2)}{q^2} = -\frac{3\pi}{\alpha}\frac{d}{dq^2}\Delta\alpha_{had}(q^2),$$

and it can be measured in e^+e^- annihilation experiments. The Adler function is related to the vacuum polarization by

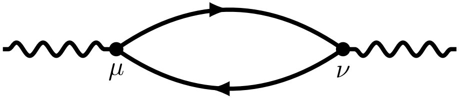
$$D(q^2) = -12\pi^2 q^2 \frac{d\Pi(q^2)}{d(q^2)}.$$

The vacuum polarization tensor can be computed by

$$\Pi_{\mu\nu}(q^2) = \int d^4x e^{iqx} \left\langle J_\mu(x) J_\nu(0) \right\rangle .$$

From Euclidean invariance and current conservation one finds

$$\Pi_{\mu\nu}(q^2) = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\Pi(q^2).$$



[1] Adler, Phys. Rev. **D** 10, 3714, 1974

Methods to compute the Adler function

Fit to $\Pi(q^2)$

Fit an ansatz to $\Pi(q^2)$, and compute the derivative of the fit function. We use the Pade ansatz

$$\Pi_{fit}(q^2) = c_0 + q^2 \left(\frac{c_1}{q^2 + c_2^2} + \frac{c_3}{q^2 + c_4^2} \right),$$

$$\frac{d}{dq^2}\Pi_{fit}(q^2) = \frac{c_1 c_2^2}{(c_2^2 + q^2)^2} + \frac{c_3 c_4^2}{(c_4^2 + q^2)^2}.$$

Numerical derivative of $\Pi(q^2)$

We use linear fits with varying ranges to approximate the derivative of $\Pi(q^2)$.

In our study we consider the CLS-ensembles given in the table below.

Label	V	β	a[fm]*	m_π[MeV]	$m_\pi L$	N_{cfg}
A5	64×32^3	5.20	0.079	312	4.0	250
E5	64×32^3	5.30	0.063	451	4.7	168
F6	96×48^3	5.30	0.063	324	5.0	217
N6	96×48^3	5.50	0.050	340	4.0	173

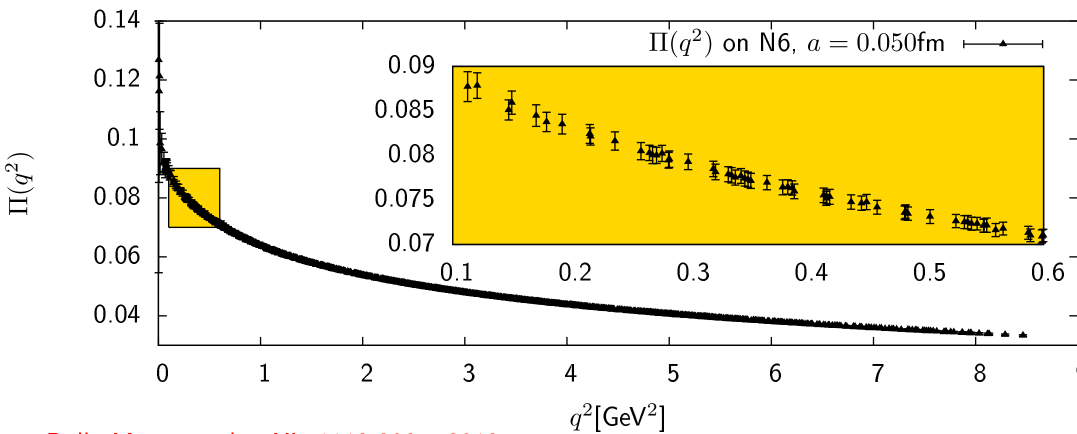
* cf. Capitani et al, arXiv:1110.6365, 2011.

Procedure I

- at each q^2 perform a linear fit

$$\Pi^{[l]}_{fit}(q^2) = a_l + b_l q^2,$$

- repeat these fits for several fit ranges $\epsilon \in [0.05, 1.0]\text{GeV}^2$,
- search for a region in ϵ where variations in b_l are small.



Della Morte et al, arXiv:1112.2894, 2012

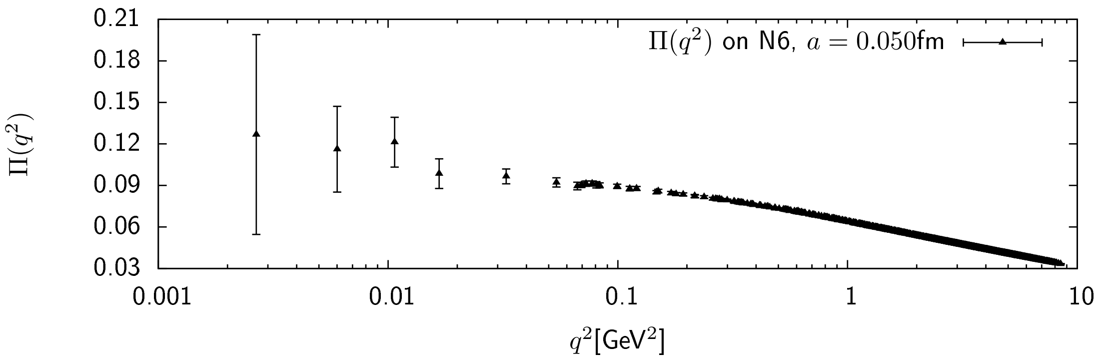
Procedures for the numerical derivative 2

Procedure II

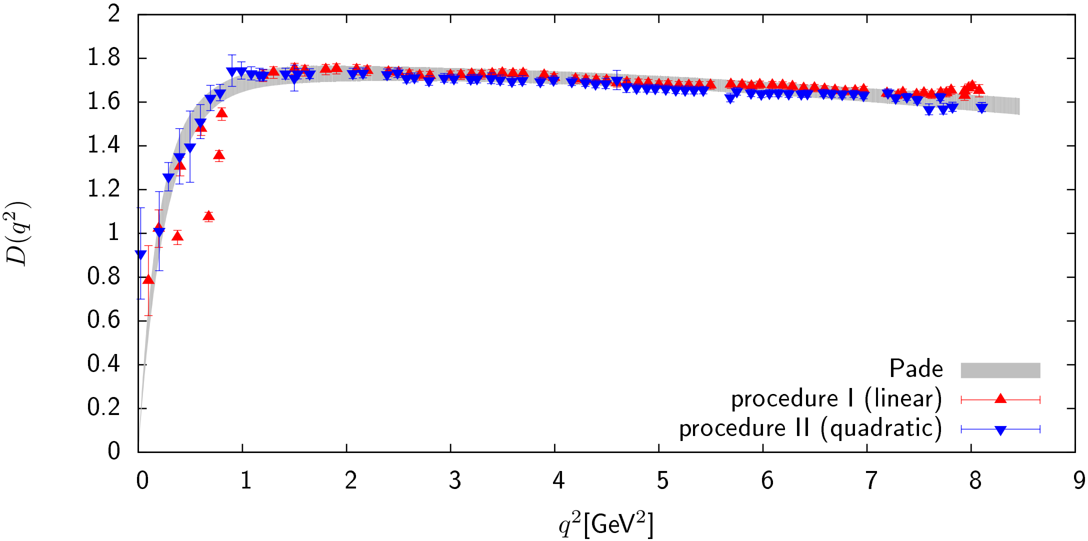
- at each q^2 we fit the two functions

$$\Pi^{[l]}_{fit}(q^2) = a_l + b_l \ln\left(q^2\right),$$
$$\Pi^{[q]}_{fit}(q^2) = a_q + b_q \ln\left(q^2\right) + c_q \left(\ln\left(q^2\right)\right)^2,$$

- repeat these fits for several fit ranges $\epsilon \in [0.05, 1.0]$GeV^2,
- apply cuts to the fits, such as removing fits with a large curvature c_q,
- from the fits that survive pick the result, where the coefficients b_l and b_q are similar.

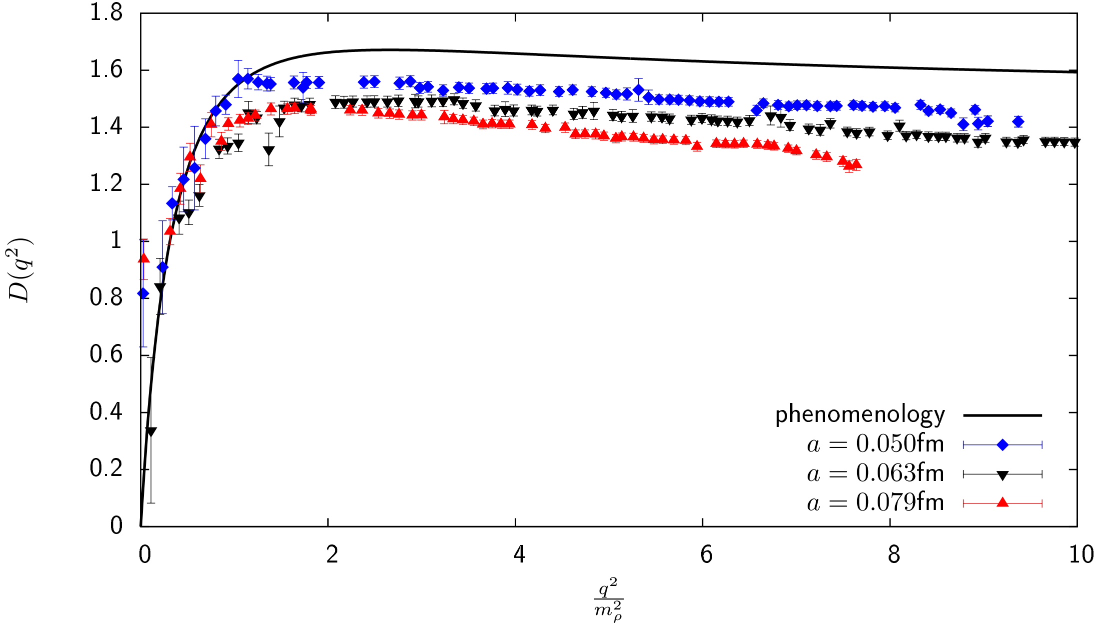


Comparison of the different methods on N6



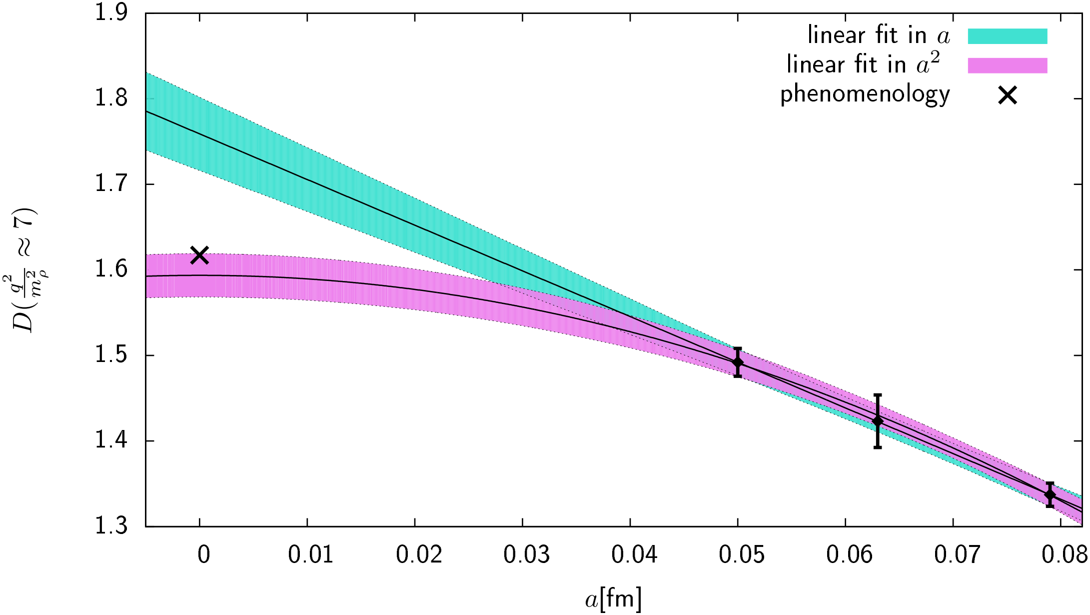
The different methods agree within errors for a large range of momentum transfers, but for very small and large values of q^2 we find deviations for procedure I.

Scaling of the Adler function 1



The phenomenological curve was provided by Meyer et al., arXiv:1306.2532, 2013.

Scaling of the Adler function 2



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Partially twisted boundary conditions 1

We use partially twisted boundary conditions, cf. Sachrajda, Villadoro, Physics Letters B, 2005, Bedaque, Physics Letters B, 2004, de Divitiis et al, Physics Letters B, 2004,

$$\Psi(x_i + L) = e^{i\Theta_i}\Psi(x_i),$$

which allow to tune the momenta to

$$\hat{p}_\mu = 2sin\left(\frac{\pi n_\mu}{L_\mu} - \frac{\Theta_\mu^{(1)} - \Theta_\mu^{(2)}}{2L_\mu}\right)$$

In simulations the twist is interpreted as a constant background field,

$$U_\mu^\Theta(x) = U_\mu(x)e^{iaB_\mu},$$

where $B_0 = 0$, and $B_i = \begin{pmatrix} B_i^{(1)} & 0 \\ 0 & B_i^{(2)} \end{pmatrix}$ with $B_i^{(j)} = \frac{\Theta_i^{(j)}}{L}$, and $\Psi(x) = \begin{pmatrix} q^{(1)} \\ q^{(2)} \end{pmatrix}$ for $N_f = 2$.

Partially twisted boundary conditions 1

From the variation of the Wilson action in the presence of twisted boundary conditions using a flavor transformation,

$$\Psi(x) \to \Psi'(x) = e^{i\alpha^a \frac{\tau^a}{2}} \Psi(x),$$
$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x) e^{-i\alpha^a \frac{\tau^a}{2}},$$

one finds that $\left[B_\mu, \frac{\tau^a}{2}\right] = 0$ is required to define a conserved current[1]. There are several ways to fulfill this condition.

(a) $B_\mu = B_\mu \mathbb{1}, \Rightarrow$ both quark fields are twisted by the same angle. Since only the difference of the twists is relevant for the momentum this would remove the effect the twisted boundary conditions have.

(b) $B_\mu = \frac{\Theta_\mu}{L} \xrightarrow[L\to\infty]{} 0$, in the infinite volume limit we recover the conserved current.

[1]cf. Aubin et al, arXiv:1307.4701, 2013 for a similar discussion.

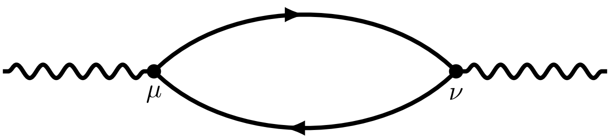
Vacuum polarization

To compute the vacuum polarization tensor

$$\Pi^{l,p}_{\mu\nu}(q^2) = \int d^4x e^{iqx} \left\langle J^{(l)}_\mu(x) J^{(ps)}_\nu(0) \right\rangle ,$$

we use the vector currents given by

$$J^{(l)}_\mu(x) = Q_f \bar{\Psi}(x)\gamma_\mu \Psi(x),$$
$$J^{(ps)}_\mu(x) = \frac{Q_f}{2}\left[\bar{\Psi}(x+\mu)U^\dagger_\mu(x)e^{-iaB_\mu}(\gamma_\mu+1)\Psi(x) + \bar{\Psi}(x)U_\mu(x)e^{iaB_\mu}(\gamma_\mu-1)\Psi(x+\mu)\right].$$

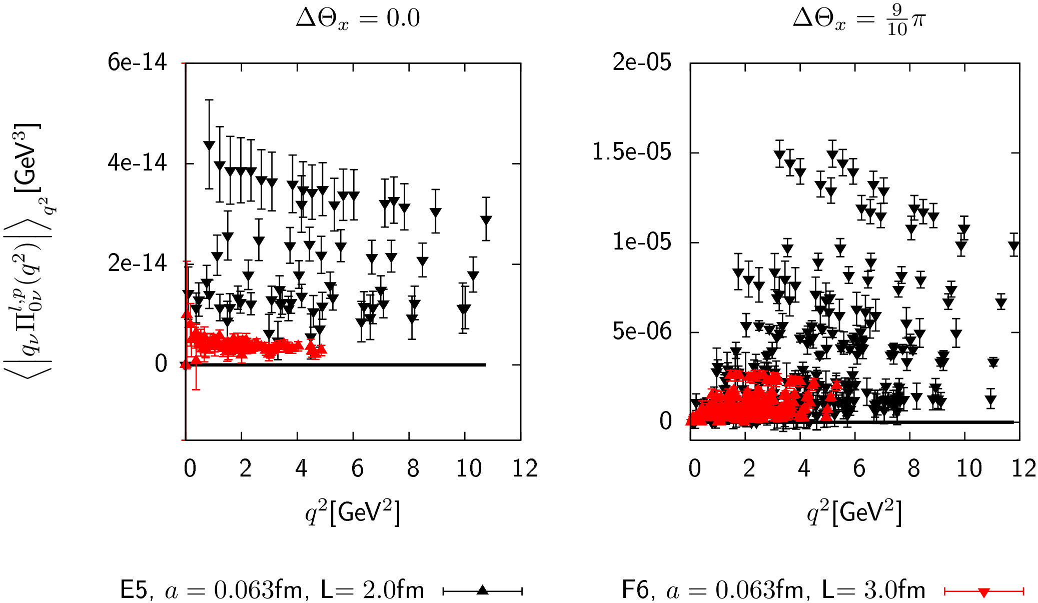


The Ward identity of the vacuum polarization in the infinite volume is given by

$$q_\nu \Pi^{l,p}_{\mu\nu}(q^2) = 0, \quad q_\mu \Pi^{l,p}_{\mu\nu}(q^2) = 0.$$

Ward identity on E5 and F6 (point-split)

As the lattice volume is increased the violation of the Ward identity vanishes for $\Delta\Theta \neq 0$.

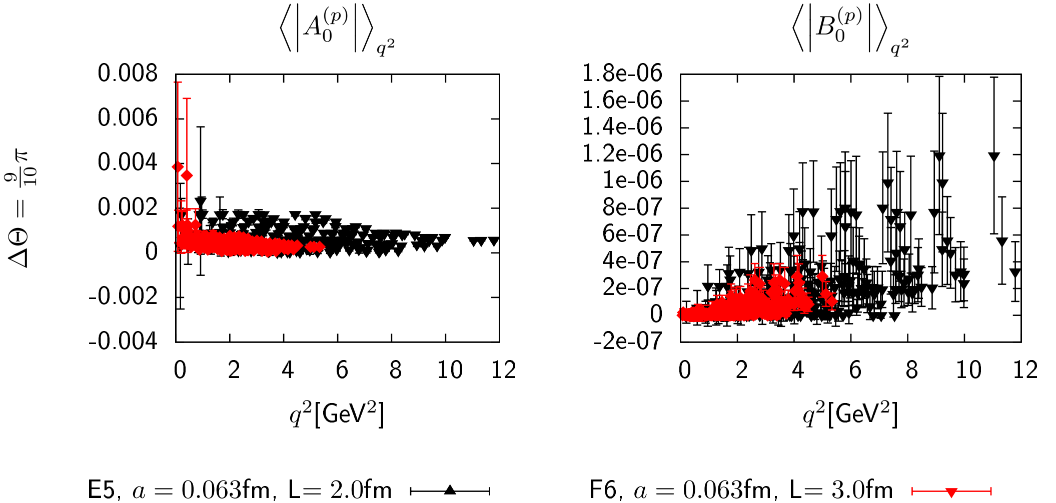


Effect of the Violation on $\Pi(q^2)$

There are different approaches to quantify the effect of the violation of the WI on $\Pi(q^2)$,

$$A_\mu^{(p)}(q^2) = \frac{\sum_\nu q_\nu \Pi_{\mu\nu}}{q_\mu \Pi_{\mu\mu}}, \quad B_\mu^{(p)}(q^2) = \frac{\left\langle \left| q_\nu \Pi_{\mu\nu}^{l,p}(q^2) \right| \right\rangle_{q^2}}{q_\mu q^2 \left\langle \Pi(q^2) \right\rangle_{q^2}}.$$

$A_\mu^{(p)}(q^2)$ is similar to a ratio used by Aubin et al, arXiv:1307.4701, 2013.



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Conclusions

Adler function

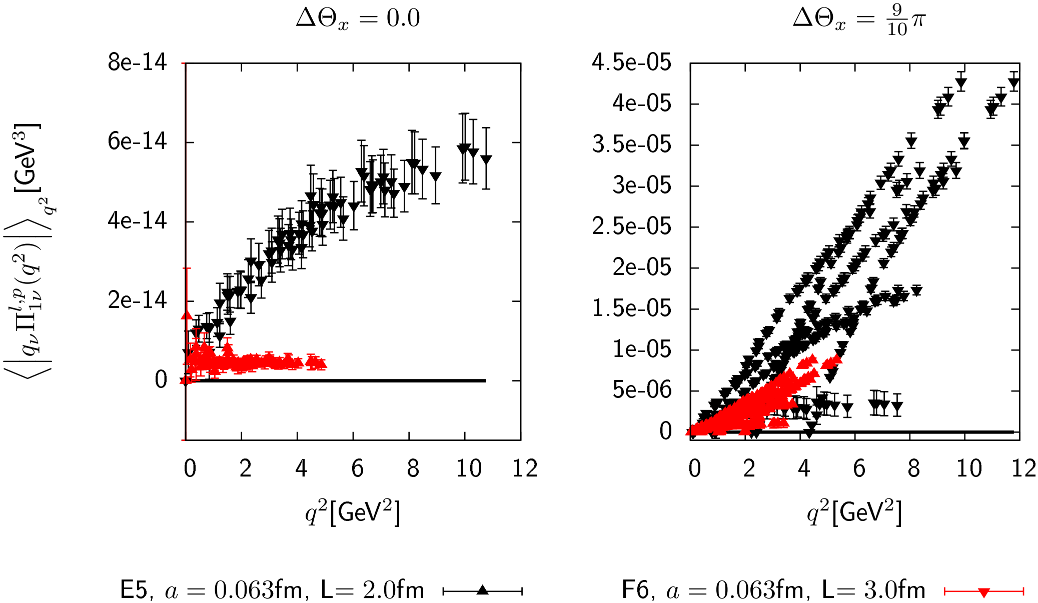
- From the vacuum polarization we can obtain the Adler function with different methods, which agree within errors for a large range of q^2.
- As we approach the continuum limit our results approach the phenomenological result.

Ward identity

- The Ward identity $q_\nu \Pi_{\mu\nu} = 0$ is not fulfilled at finite size.
- For our largest twist angle $\Theta = \frac{9\pi}{10}$ the violation of the Ward identity for the point-split current is of $\mathcal{O}\left(10^{-5}\right)$.
- The effect the violation has on the vacuum polarization is below the current sensitivity and thus negligible.

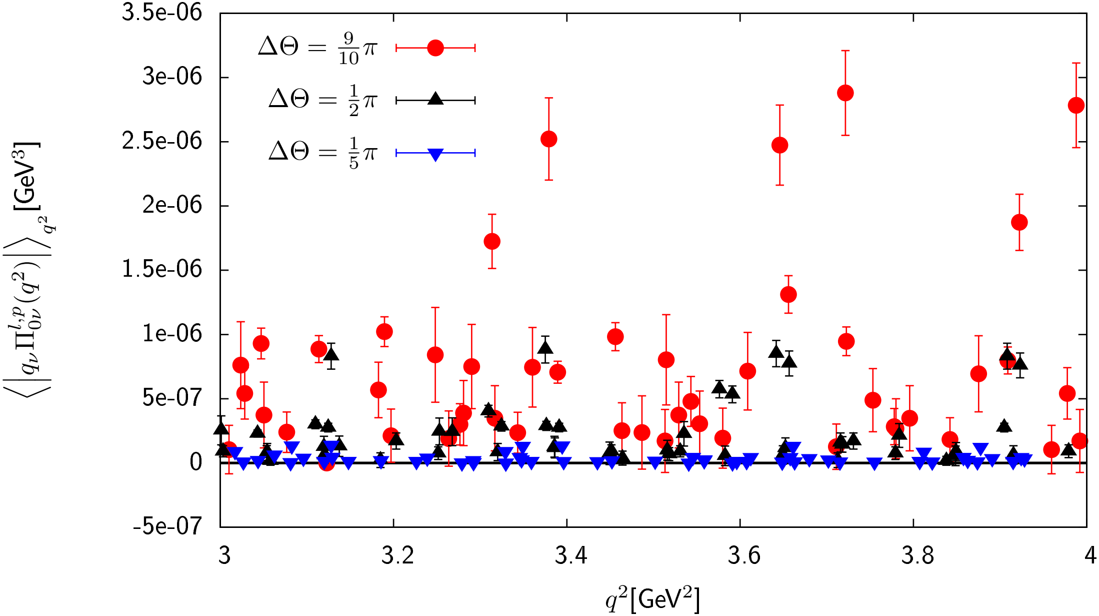
Thank you for your attention.

Ward identity on E5 and F6 (point-split)[X]



Ward identity on N6, $\beta = 5.50$, $a = 0.050 fm$ (point-split)

For larger twist angles $\Delta\Theta$ the violation of the Ward identity becomes more severe.



Ward identity on N6, F6, and A5 (local)

As the continuum limit is approached the violation of the Ward identity due to the local current diminishes.