Leading-order hadronic contribution to the anomalous magnetic moment of the muon from $N_f = 2 + 1 + 1$ twisted mass fermions

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Why the muon’s anomalous magnetic moment?

- The anomalous magnetic moment of the muon, $a_\mu$, can be measured very precisely: [B. Lee Roberts, Chinese Phys. C 34, 2010]

\[
a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}
\]

\[
a_\mu^{\text{SM}} = 116591828(49) \times 10^{-11}
\]

[Hagiwara et al., J. Phys. G38, 2011]

There is a $\approx 3\sigma$ discrepancy between $a_\mu^{\text{exp}}$ and $a_\mu^{\text{SM}}$:

\[
a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 261(80) \times 10^{-11}
\]
Charm quark necessary to reach required precision


\[ a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = 261(80) \times 10^{-11} \]

  \[ a_{\mu}^{\text{hvp,c}} = 144(1) \times 10^{-11} \]
  \[ a_{\mu}^{\text{hlbl}} = 105(26) \times 10^{-11} \]
- And also to electroweak contribution [Jegerlehner, Nyffeler, Phys. Rept. 477, 2009]
  \[ a_{\mu}^{\text{EW}} = 153(2) \times 10^{-11} \]
Leading hadronic contribution $a_{\mu}^{\text{hvp}}$

\[ a_{\mu}^{\text{QCD}} = a_{\mu}^{\text{lo,hvp}} + a_{\mu}^{\text{ho,hvp}} + a_{\mu}^{\text{lbl}} \]

- can be computed directly in Euclidean space-time [T. Blum, PRL 91, 2003]

\[ a_{\mu}^{\text{hvp}} = \alpha^2 \int_0^{\infty} \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{m_{\mu}^2} \right) \Pi_R(Q^2) \]

where $\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$

- main ingredient: hadronic vacuum polarisation tensor

\[ \Pi_{\mu\nu}(Q) = \int d^4 x \, e^{iQ \cdot (x-y)} \langle J_{\mu}^{\text{em}}(x) J_{\nu}^{\text{em}}(y) \rangle = (Q_\mu Q_\nu - Q^2 g_{\mu\nu}) \Pi(Q^2) \]

with

\[ J_{\mu}^{\text{em}}(x) = \frac{2}{3} \bar{u}(x) \gamma_\mu u(x) - \frac{1}{3} \bar{d}(x) \gamma_\mu d(x) + \frac{2}{3} \bar{c}(x) \gamma_\mu c(x) - \frac{1}{3} \bar{s}(x) \gamma_\mu s(x) \]
Mixed-action set-up

- configurations generated by ETMC [Baron et al., JHEP 1006, 2010]

\[ S_F[\chi, \chi^\ast, U] = \sum_x \chi^\ast(x) \left[ D_W + m_0 + i\mu_q\gamma_5\tau^3 \right] \chi(x) \]


\[ S_F[\chi_h, \chi_h^\ast, U] = \sum_x \chi_h^\ast(x) \left[ D_W + m_0 + i\mu_\sigma\gamma_5\tau^1 + \mu_\delta\tau^3 \right] \chi_h(x) \]

- heavy valence quarks: Osterwalder-Seiler action [Frezzotti, Rossi, JHEP 0410, 2004]

\[ S_F[\chi_h, \chi_h^\ast, U] = \sum_x \chi_h^\ast(x) \left[ D_W + m_0 + i\left( \begin{array}{cc} \mu_c & 0 \\ 0 & -\mu_s \end{array} \right) \gamma_5 \right] \chi_h(x) \]

- tune bare mass parameters \( \mu_{c/s} \) such that physical kaon and D-meson masses are reproduced
The $N_f = 2 + 1 + 1$ ensembles

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>$\beta$</th>
<th>$a$[fm]</th>
<th>$L^3 \times T$</th>
<th>$m_{PS}$[MeV]</th>
<th>$L$[fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>D15.48</td>
<td>2.10</td>
<td>0.061</td>
<td>$48^3 \times 96$</td>
<td>227</td>
<td>2.9</td>
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<td>D30.48</td>
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<td>$48^3 \times 96$</td>
<td>318</td>
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<tr>
<td>D45.32sc</td>
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<tr>
<td>B25.32t</td>
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<td>0.078</td>
<td>$32^3 \times 64$</td>
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<td>319</td>
<td>2.5</td>
</tr>
<tr>
<td>B35.48</td>
<td>1.95</td>
<td>0.078</td>
<td>$48^3 \times 96$</td>
<td>314</td>
<td>3.7</td>
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<td>0.078</td>
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<tr>
<td>A30.32</td>
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<td>0.086</td>
<td>$32^3 \times 64$</td>
<td>283</td>
<td>2.8</td>
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<tr>
<td>A40.32</td>
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<td>0.086</td>
<td>$32^3 \times 64$</td>
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<td>2.8</td>
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<tr>
<td>A50.32</td>
<td>1.90</td>
<td>0.086</td>
<td>$32^3 \times 64$</td>
<td>361</td>
<td>2.8</td>
</tr>
</tbody>
</table>
First $N_f = 2$ configurations at the physical point

More details: Talk by Bartosz Kostrzewa, Monday, 16:50

- again use Iwasaki action in gauge sector
- add clover-term to twisted mass action for non-degenerate fermion doublet

$$S_F[\chi, \bar{\chi}, U] = \sum_x \bar{\chi}(x) \left[ D_W + m_0 + i\mu q\gamma_5\tau^3 \right] \chi(x)$$

$$+ c_{SW} \sum_x \bar{\chi}(x) \left[ i\frac{1}{4} \sigma_{\mu\nu} F_{\mu\nu} \right] \chi(x)$$

- very preliminary parameters of first ensemble:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$c_{SW}$</th>
<th>$a[\text{fm}]$</th>
<th>$L^3 \times T$</th>
<th>$m_{PS}[\text{MeV}]$</th>
<th>$L[\text{fm}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.10</td>
<td>1.57551</td>
<td>0.096</td>
<td>$48^3 \times 96$</td>
<td>128</td>
<td>4.6</td>
</tr>
</tbody>
</table>
How the observables are determined

- use conserved (point-split) vector current

\[ J^C_\mu(x) = \frac{1}{2} \left( \bar{\chi}(x + \hat{\mu})(1 + \gamma_\mu)U^\dagger_\mu(x)Q_{el}\chi(x) 
- \bar{\chi}(x)(1 - \gamma_\mu)U_\mu(x)Q_{el}\chi(x + \hat{\mu}) \right) \]

where \( Q_{el} = \text{diag}(\frac{2}{3}, -\frac{1}{3}) \)

- use redefinition \cite{Feng, Jansen, Petschlies, Renner, PRL 107, 2011}

\[ a^{hvp}_\mu = \alpha^2 \int_0^{\infty} \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{H^2} \frac{H^2_{phys}}{m^2_\mu} \right) \Pi_R(Q^2) \]

which goes to \( a^{hvp}_\mu \) for \( m_{PS} \to m_\pi \), i.e. when \( H \to H_{phys} \)

- effectively, redefinition of muon mass

\[ m_{\mu} = m_{\mu} \cdot \frac{H}{H_{phys}} \]

- in the following will always use \( H = m_V \) - \( \rho \)-meson mass
Fitting the hadronic vacuum polarisation function

- have $\Pi(\hat{Q}^2)$ depending on discrete momenta
- to obtain smooth function fit this for each flavour to

$$\Pi(Q^2) = (1 - \theta(Q^2 - Q^2_{\text{match}}))\Pi_{\text{low}}(Q^2) + \theta(Q^2 - Q^2_{\text{match}})\Pi_{\text{high}}(Q^2)$$

with

$$\Pi_{\text{low}}(Q^2) = \sum_{i=1}^{M} g_i^2 \frac{m_i^2}{Q^2 + m_i^2} + \sum_{j=0}^{N-1} a_j(Q^2)^j$$

and

$$\Pi_{\text{high}}(Q^2) = \sum_{k=0}^{C-1} c_k(Q^2)^k + \left(\sum_{l=0}^{B-1} b_l(Q^2)^l\right) \cdot \log(Q^2)$$

- different matching conditions and functions possible
Light quark contribution on $N_f = 2 + 1 + 1$ sea

\[ H = m_{\nu} \]

- $N_f = 2 + 1 + 1$ result: $a_{\mu, ud}^{hvp} = 5.67(11) \cdot 10^{-8}$
- $N_f = 2$ result: $a_{\mu, ud}^{hvp} = 5.72(16) \cdot 10^{-8}$

[Feng, Jansen, Petschlies, Renner, PRL 107, 2011]
Light quark contribution on $N_f = 2 + 1 + 1$ sea

Comparing to preliminary $N_f = 2$ result at physical pion mass

$H = m_V$

$H = 1$

- $N_f = 2 + 1 + 1$ result: $a_{\mu, ud}^{\text{hvp}} = 5.67(11) \cdot 10^{-8}$
- new $N_f = 2$ result at physical point: $a_{\mu, ud}^{\text{hvp}} = 5.55(70) \cdot 10^{-8}$
Adding the strange quark in the valence sector

- For strange quark lattice artefacts have to be taken into account

\[ a^{\text{hvp}}_{\mu,uds}(m_{\text{PS}}, a) = A + B \, m_{\text{PS}}^2 + C \, a^2 \]

with fit parameters \( A, B, C \)

Light sector: cannot discriminate \( a^2 \) effects
Three-flavour contribution on $N_f = 2 + 1 + 1$ sea

- $N_f = 2 + 1 + 1$ result: $a_{\mu,uds}^{\text{hvp}} = 6.55(21) \cdot 10^{-8}$
- three-flavour result extracted from dispersive analysis of $[\text{Jegerlehner, Szafron, Eur. Phys. J C71, 2011}]$: $a_{\mu,uds}^{\text{hvp}} = 6.79(05) \cdot 10^{-8}$
The four-flavour contribution on $N_f = 2 + 1 + 1$ sea

$H = m_V$

- $N_f = 2 + 1 + 1$ result: $a_{\mu}^{\text{hvp}} = 6.74(21) \cdot 10^{-8}$
- result from dispersive analysis: [Jegerlehner, Szafron, Eur. Phys. J C71, 2011] $a_{\mu}^{\text{hvp}} = 6.91(05) \cdot 10^{-8}$
Systematic uncertainties

- from choosing fit ranges for vector mesons:
  \[ \Delta_V = 0.13 \cdot 10^{-8} \]

- from choosing different number of terms in fit function:
  \[ \Delta_{MNBC} = 0.12 \cdot 10^{-8} \]

- found to be negligible: \( m_{PSL} > 3.8 \), \( m_{PS} < 400 \text{ MeV} \), varying matching momentum between \([1 \text{ GeV}^2, 3 \text{ GeV}^2]\)
- not quantified yet: disconnected contributions, wrong sea quark masses
- preliminary final result:
  \[ a_{\mu}^{\text{hvp}} = 6.74(21)(18) \cdot 10^{-8} \]
Comparison of results for $a_h^{\mu}$

- $a_h^{\mu}$ in $10^{-8}$ for different numbers of valence quarks:

<table>
<thead>
<tr>
<th></th>
<th>u,d</th>
<th>u, d, s</th>
<th>u, d, s, c</th>
</tr>
</thead>
<tbody>
<tr>
<td>this work</td>
<td>5.67(11)</td>
<td>6.55(21)</td>
<td>6.74(27)</td>
</tr>
<tr>
<td>ETMC 2011</td>
<td>5.72(16)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mainz 2011</td>
<td>5.46(66)</td>
<td>6.18(64)</td>
<td>-</td>
</tr>
<tr>
<td>RBC-UKQCD 2011</td>
<td>-</td>
<td>6.41(33)</td>
<td>-</td>
</tr>
<tr>
<td>HLS estimate 2011</td>
<td>5.59(04)</td>
<td>6.71(05)</td>
<td>6.83(05)</td>
</tr>
<tr>
<td>(dispersive analysis + flavour weighting)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

ETMC 2011: [Feng, Jansen, Petschlies, Renner, PRL 107, 2011]
Mainz 2011: [Della Morte, Jäger, Jüttner, Wittig, JHEP 1203, 2012]
Summary

- First $N_f = 2 + 1 + 1$ lattice calculation of $a_{\mu}^{hvp}$ gives compatible result with dispersive analyses.

- Modified method [Feng, Jansen, Petschlies, Renner, PRL 107, 2011] works for $N_f = 2 + 1 + 1$ computation.

- Chiral extrapolation in light sector to be checked with computation at physical pion mass.
Outlook

- improve data by more statistics, especially in heavy sector, and all-mode-averaging
- use Padé approximants
- disconnected contributions
- more $N_f = 2$ configurations at physical point
- $N_f = 2 + 1 + 1$ configurations at physical point, probably several lattice spacings needed
- include isospin breaking effects
- different observables: $\Delta \alpha_{\text{QED}}^{\text{hvp}}$, Adler function, weak mixing angle, S-parameter
- light-by-light scattering
Why it works

- **redefinition** [Feng, Jansen, Petschlies, Renner, PRL 107, 2011]

\[
a_{\mu}^{\text{hvp}} = \alpha_0^2 \int_0^{\infty} \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{H^2} \frac{H^2_{\text{phys}}}{m_{\mu}^2} \right) \Pi_R(Q^2)
\]

which goes to \( a_{\mu}^{\text{hvp}} \) for \( m_{PS} \rightarrow m_\pi \), i.e. when \( H \rightarrow H_{\text{phys}} \)

- **effectively**, redefinition of muon mass

\[
m_{\mu} = m_{\mu} \cdot \frac{H}{H_{\text{phys}}}
\]

- **leading vector meson contribution**

\[
a_{\mu}^{\text{hvp}} \propto \alpha^2 g_V^2 \frac{m_{\mu}^2}{m_V^2}
\]

\( \Rightarrow \) strong dependence on \( m_{PS} \) via \( m_V \)
Comparison with Padé approximants

- vacuum polarisation integrated up to $Q^2_{\text{max}} = 1.5 \, \text{GeV}^2$:

will use Padé approximants when chiral extrapolation no longer needed
Example for standard fit

\[ \beta = 1.95, \frac{L}{a} = 32, \mu_{\text{light}} = 0.0025 \]