1

Quantum Mechanics à la Langevin and Supersymmetry

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Outline

QM in the Stochastic Formulation

$$\frac{\partial\phi(t,\tau)}{\partial t} = -\frac{\partial U}{\partial\phi} + \eta(t,\tau) \stackrel{t\to\infty}{\longrightarrow} \frac{\partial U}{\partial\phi(\tau)} = \eta(\tau)$$

$$\frac{\partial U}{\partial\phi(\tau)} = \frac{\partial\phi}{\partial\tau} + \frac{\partial W}{\partial\phi(\tau)}$$

$$Z \equiv \int [d\eta(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\eta(\tau)^2}\, \delta\left(\frac{d\phi(\tau)}{d\tau} + \frac{\partial W}{\partial\phi(\tau)} - \eta(\tau)\right) =$$

$$\int [d\phi(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\left(\frac{d\phi}{d\tau} + \frac{\partial W}{\partial\phi(\tau)}\right)^2} \left|\det\left(\delta(\tau-\tau')\frac{d}{d\tau} + \frac{\partial^2 W}{\partial\phi(\tau)\partial\phi(\tau')}\right)\right|$$

Grassmann variables for a local action

Assumption : W is ultra local :

$$\frac{\partial^2 W}{\partial\phi(\tau)\phi(\tau')} = \delta(\tau-\tau')\frac{\partial^2 W}{\partial\phi(\tau)^2}$$

Then

$$\left|\det\left(\delta(\tau-\tau')\frac{d}{d\tau} + \frac{\partial^2 W}{\partial\phi(\tau)\partial\phi(\tau')}\right)\right| =$$
$$|\det\delta(\tau-\tau')|\left|\det\left(\frac{\partial}{\partial\tau} + \frac{\partial^2 W}{\partial\phi(\tau)^2}\right)\right|$$

Langevin and Grassmann

$$\left|\det\left(\frac{\partial}{\partial\tau}+\frac{\partial^2 W}{\partial\phi(\tau)^2}\right)\right| =$$
$$\int [d\psi_1(\tau)d\psi_2(\tau)]\, e^{\int d\tau\, \frac{1}{2}\psi_\alpha\varepsilon^{\alpha\beta}\left(\frac{d}{d\tau}+\frac{\partial^2 W}{\partial\phi(\tau)^2}\right)\psi_\beta(\tau)}$$

Enter SUSY ?

The classical action

$$S = \int d\tau\, d\tau' \left[\frac{1}{2} \left(\frac{\partial U}{\partial \phi(\tau)} \right)^2 \delta(\tau - \tau') - \frac{1}{2} \psi_\alpha(\tau) \varepsilon^{\alpha\beta} \frac{\partial^2 U}{\partial \phi(\tau) \partial \phi(\tau')} \psi_\beta(\tau') \right]$$

is invariant under

$$\delta\phi(\tau) = \zeta_\alpha \varepsilon^{\alpha\beta} \psi_\beta(\tau)$$
$$\delta\psi_\alpha = \zeta_\alpha \frac{\partial U}{\partial \phi(\tau)}$$

with ζ_α Grassmann variables–*if* U is ultra–local. What happens if U local ?

. . .Auxiliary

We can linearize this transformation, by introducing an auxiliary field :

$$e^{-\frac{1}{2}\left(\frac{\partial U}{\partial \phi}\right)^2} = \int d(\mathrm{i}F)\, e^{-\frac{1}{2}F^2 + F\frac{\partial U}{\partial \phi}}$$

Its equation of motion is

$$F = \frac{\partial U}{\partial \phi}$$

which implies that

$$F = \eta$$

It's drawn from a Gaussian, so its correlation functions should satisfy Wick's theorem.

If U is ultra local, then the transformation

$$\delta\phi = \zeta_\alpha \varepsilon^{\alpha\beta}\psi_\beta$$
$$\delta\psi_\alpha = \zeta_\alpha F$$
$$\delta F = 0$$

is the zero–dimensional reduction of $\mathcal{N} = 1$ SUSY. This isn't true for local U.

Elusive SUSY

If U is local, we need, apparently *two* fermions, ψ *and* χ

$$S = \int d\tau \left[-\frac{1}{2}F^2 + F\left(\frac{\partial\phi}{\partial\tau} + W'(\phi)\right) \right. \\ \left. -\psi_\alpha \varepsilon^{\alpha\beta}\left(\frac{\partial\chi_\beta}{\partial\tau} + W''(\phi)\chi_\beta\right)\right]$$

to realize the SUSY algebra(de Alfaro, Fubini, Furlan, Veneziano, 1985), i.e. so that

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}\frac{\partial}{\partial\tau}$$

Mismatch of degrees of freedom : 4 "fermions" vs. 2 "bosons" ? ? It's just an illusion...

Elusive SUSY

We consider ψ and χ as two "flavors" and write $\psi = \Psi_1$, $\chi = \Psi_2$. The fermionic action can then be written as

$$S_{\text{fermionic}} = \int d\tau \left[\Psi_I K^{IJ} \frac{\partial}{\partial \tau} \Psi_J + \Psi_I L^{IJ} W''(\phi) \Psi_J \right]$$

with $K = -\mathrm{i}\sigma_2/2$ and $L = \sigma_1/2$. "Triplet" structure.

Who's propagating ?

Diagonalize K^{IJ} by a, global, unitary, transformation. Then notice that the kinetic term becomes a total derivative :

$$X_{\alpha,I}\lambda^I\varepsilon^{\alpha\beta}\delta^{IJ}\frac{\partial}{\partial\tau}X_{\beta,J}=\frac{\partial}{\partial\tau}\left(\frac{\lambda^I}{2}X_{\alpha,I}\delta^{IJ}\varepsilon^{\alpha\beta}X_{\beta,J}\right)$$

Only dipoles propagate ! So the fermions can be integrated out and leave a "Polyakov loop" behind (for periodic bc for scalar) :

$$S_{\mathrm{fermionic}}=\int d\tau\,\log W''(\phi)+\delta S_{\mathrm{boundary,f}}$$

Focus on

$$W=\frac{m^2}{2}\phi^2+\frac{\lambda}{4!}\phi^4$$

with $m^2>0,\lambda>0$, to avoid zeromodes.

A bosonic lattice action for SUSY QM

We can write the lattice action for the scalar as follows :

$$S_{\rm latt} = m_{\rm latt}^2 gs \sum_{n=0}^{N-1} \left\{ -\Phi_n \Phi_{n+1} + \Phi_n^2 + \frac{m_{\rm latt}^4}{2} \left(\Phi_n + \frac{\Phi_n^3}{6} \right)^2 - m_{\rm latt}^4 g^2 s \log \left[\frac{1}{gs} \left(1 + \frac{\Phi_n^2}{2} \right) \right] \right\}$$

with $m_{\rm latt}^2 = m^2 a$, $\lambda_{\rm latt} = \lambda a^2$. We note that $g \equiv \lambda_{\rm latt}/m_{\rm latt}^4$ is scale invariant and $a(m_{\rm latt}^2/\lambda_{\rm latt}) = m^2/\lambda \equiv s$. The lattice field $\phi_n = \Phi_n (a m_{\rm latt}^2/\lambda_{\rm latt})^{1/2}$.

The auxiliary field

The observables are the correlation functions of the auxiliary field :

$$\mathfrak{F}_n = \frac{1}{2}\left(\Phi_{n+1} - \Phi_{n-1}\right) + m_{\mathrm{latt}}^2\left(\Phi_n + \frac{\Phi_n^3}{6}\right)$$

Is the auxiliary field an ultra–local Gaussian ?

- 1–point function vanishes, already on the lattice :

$$\langle\mathfrak{F}_n\rangle = \frac{1}{2}\left(\langle\Phi_{n+1}\rangle - \langle\Phi_{n-1}\rangle\right) + m_{\mathrm{latt}}^2\left(\langle\Phi_n\rangle + \frac{\langle\Phi_n^3\rangle}{6}\right) = 0$$

- 2–point function, on the lattice, not obviously ultra–local. We would like to check that

$$\langle\mathfrak{F}_n\mathfrak{F}_{n'}\rangle = \left\langle\mathfrak{F}_0\mathfrak{F}_{|n-n'|}\right\rangle \propto \delta_{|n-n'|,0}$$

- Connected 4–point function vanishes. Easy part, check that the *Binder cumulant*

$$\left\langle\mathfrak{F}_n^4\right\rangle - 3\left\langle\mathfrak{F}_n^2\right\rangle^2 = 0$$

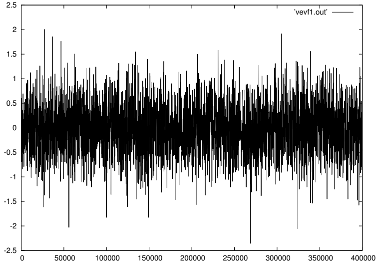
Does the scalar field become a Gaussian too ?

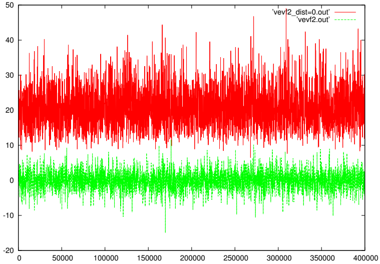
We should check that the scalar is *not* drawn from a Gaussian, in the scaling limit !
But, for consistency,

$$\langle \Phi_n \rangle = 0$$

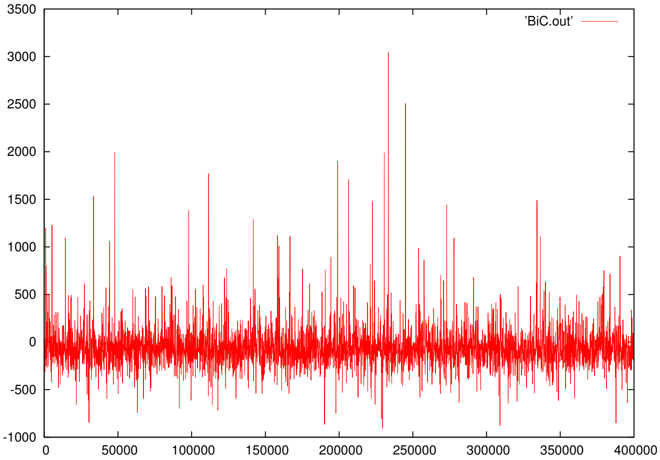
to make sure that quantum effects don't lead to a totally different potential.

Sample data : Auxiliary Field

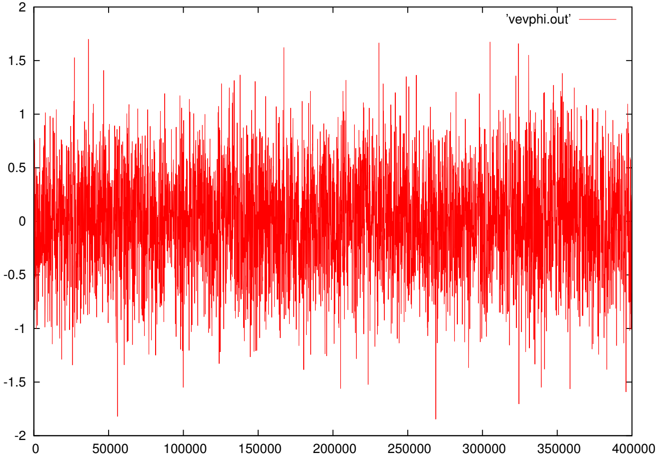




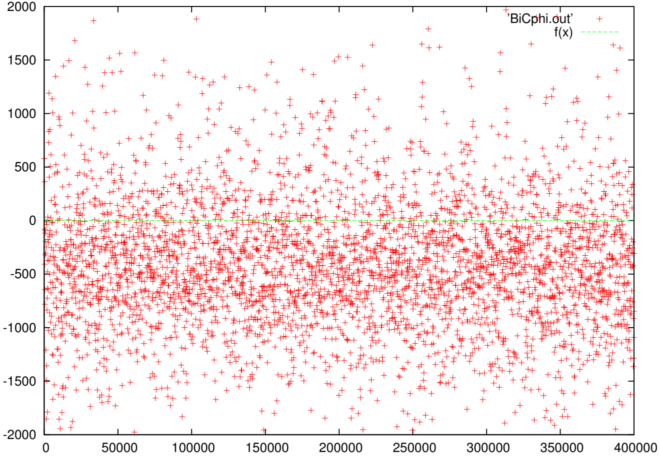
Sample date : Auxiliary Field



Sample data : The scalar field



Sample data : The scalar field



Conclusions–Perspectives

- In quantum mechanics the superpartners to the "scalar fields"–that label the position–are, in fact, part of the model : they are the worldline Grassmann variables, that express the quantum fluctuations in a local manner in the action.
- In quantum mechanics the "fermions" do note propagate as such–they remain confined in dipoles, whose propagation is controlled by the scalar. This is due to the triviality of the "Dirac algebra" in $d = 1$.
- It may be possible to realize experimental measurements of the appropriate correlation functions with trapped ions and, perhaps, probe superspace in this way.
- Considering coupling to the electromagnetic field may lead to vector superfields.