Cyclic Leibniz rule: a formulation of supersymmetry on lattice

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Cyclic Leibniz rule: a formulation of supersymmetry on lattice
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For the purpose of constructing supersymmetric(SUSY) theories on lattice, we propose a new type relation on lattice -cyclic Leibniz rule(CLR)- which is slightly different from an ordinary Leibniz rule. Actually, we find that the CLR can enlarge the number of SUSYs from N=1 to N=2 in the quantum-mechanical model.

Obvious incompatiblity between
Lattice theories and (super-)Poincare Symmetry
Nevertherless,
the nonperturbative analysis is very attractive.

What expectations for supersymmetry on lattice?

(i) exact mass degeneracy between fermion and boson
(ii) stablity of theory against any quantum fluctuation

such as non-renormalization theorem, ...

If we could obtain an exact superalgebra on lattice,

$$\{Q, \bar{Q}\} = 2\sigma_\mu P_\mu (= 2i\sigma_\mu \partial_\mu),$$

these expectations would be realized.

Our Purpose 2

- Free theory case, the expectation can be fully achieved, if $\Delta^T = -\Delta$.
- Interacting theory $\cdots$ Difficult

We set

$$(\Delta\phi)_m \equiv \sum_n \Delta_{mn}\phi_n,\ \{\phi,\phi\}^M_\ell \equiv \sum_{mn} M_{\ell mn}\phi_m\phi_n$$

No-go theorem on Leibniz rule on lattice by us

JHEP0805(2008)057(arXiv:0803.3121)

Two simple assumptions $\rightarrow$ Holomorphy

(i) Locality(L)

$$\Delta_{mn} < C_1 \exp\left(-C_2|m-n|\right), \text{with } |m-n| \rightarrow \infty$$

$C_1, C_2 > 0$. Similarily, for M.

(ii) Translational invariance(T)

$$\Delta_{mn} = \Delta(m-n),\ M_{\ell mn} = M(\ell - m, \ell - n)$$

Our Purpose 3

Holomorphic functions on $1-\epsilon < |w,z| < 1+\epsilon'$

$$\hat{\Delta}(w) \equiv \sum_{m} w^{m-n}\Delta_{m,n} = \sum_{m-n} w^{m-n}\Delta_{m-n,0},$$

$$\hat{M}(w,z) \equiv \sum_{\ell,m} w^{k-\ell} z^{k-m} M_{k\ell m}$$

By these holomorphic functions, the Leibniz rule $\Delta\{\phi,\psi\}^M = \{\Delta\phi,\psi\}^M + \{\phi,\Delta\psi\}^M$ is written as

$$\rightarrow \hat{\Delta}(wz)\hat{M}(w,z) = \hat{\Delta}(w)\hat{M}(w,z) + \hat{\Delta}(z)\hat{M}(w,z)$$

The unique solution for this with $w = \exp(ipa)$ is nonholomorphic

$$\rightarrow \hat{\Delta}(w) \sim \log w \sim p \quad (\text{SLAC type;} \quad \text{nonlocal})$$

No-go theorem : there is no lattice theory keeping locality, translational invariance(T) and Leibniz rule(LR).

In other words, T and LR on lattice enforce the non-holomorphic i.e. nonlocal property.

$\boxed{\text{Our purpose}}$

to find key notion instead of Leibniz rule on lattice

in addition to holomorphy

The answer is *Cyclic Leibniz Rule*.

Contents

To construct interacting SCQM 1

From now, we construct $N = 2, \; D = 1$ model.

Supersymmetric Complex Quantum Mechanics

Dynamical degree of freedom

$$\phi_n, \; \bar{\phi}_n, \; \chi_{\pm n}, \; \bar{\chi}_{\pm n}, \; F_n, \; \bar{F}_n$$

Four kinds of field product rules

- Product rule for ϕ, ψ

$$\{\phi, \psi\}^M_k \equiv \sum_{mn} M_{kmn}\phi_m\psi_n, \;\; \{\phi, \psi\}^N_k \equiv \sum_{mn} N_{kmn}\phi_m\psi_n$$

- Product rule for $\bar{\phi}, \bar{\psi}$

$$\{\bar{\phi}, \bar{\psi}\}^{\bar{M}}_k \equiv \sum_{mn} \bar{M}_{kmn}\bar{\phi}_m\bar{\psi}_n, \;\; \{\bar{\phi}, \bar{\psi}\}^{\bar{N}}_k \equiv \sum_{mn} \bar{N}_{kmn}\bar{\phi}_m\bar{\psi}_n$$

To construct interacting SCQM 2

$N = 2$ SUSY transformation for this $D = 1$ model

$$
\begin{aligned}
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\phi_n &= \bar{\epsilon}_-\chi_{+n} + \chi_{-n}\epsilon_+ \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\chi_{+n} &= i\epsilon_-(\Delta\phi)_n - \epsilon_+F_n \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\chi_{-n} &= -i\bar{\epsilon}_+(\Delta\phi)_n - \bar{\epsilon}_-F_n \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}F_n &= -i\epsilon_-(\Delta\chi_-)_n - i\bar{\epsilon}_+(\Delta\chi_+)_n \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\bar{\phi}_n &= \bar{\chi}_{+n}\epsilon_- + \bar{\epsilon}_+\bar{\chi}_{-n} \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\bar{\chi}_{+n} &= -i\bar{\epsilon}_-(\Delta\bar{\phi})_n - \bar{\epsilon}_+\bar{F}_n \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\bar{\chi}_{-n} &= i\epsilon_+(\Delta\bar{\phi})_n - \epsilon_-\bar{F}_n \\
\delta_{\epsilon_\pm,\bar{\epsilon}_\pm}\bar{F}_n &= -i\epsilon_+(\Delta\bar{\chi}_+)_n - i\bar{\epsilon}_-(\Delta\bar{\chi}_-)_n
\end{aligned}
$$

Free action $\rightarrow$ invariant when $\Delta_{mn} = -\Delta_{nm}$ or $\Delta = -\Delta^T$

$$S_0 = (\Delta\bar{\phi}, \Delta\phi) + i(\bar{\chi}_+, \Delta\chi_+) + i(\bar{\chi}_-, \Delta\chi_-) + (\bar{F}, F)$$

An inner product(sum over lattice)

$$(A, B) = \sum_n A_n B_n$$

Interaction terms in the action

$$\begin{aligned} S_{\text{int}} \;&=\; ig_2\Big((F, \{\phi, \phi\}^M) + 2(\phi, \{\chi_+, \chi_-\}^N)\Big) \\ &\quad -ig_2^*\Big((\bar{F}, \{\bar{\phi}, \bar{\phi}\}^{\bar{M}}) + 2(\bar{\phi}, \{\bar{\chi}_-, \bar{\chi}_+\}^{\bar{N}})\Big) \end{aligned}$$

where $\{A, B\}^M = \{B, A\}^M$ and $\{A, B\}^{\bar{M}} = \{B, A\}^{\bar{M}}$.

no-go theorem again

SUSY-invariance conditions: $\delta_{\epsilon_\pm, \bar{\epsilon}_\pm} S_{\text{int}} = 0 \rightarrow$

$$(\Delta\chi_-, \{\phi, \phi\}^M) = -2(\phi, \{\Delta\phi, \chi_-\}^N) = -2(\chi_-, \{\Delta\phi, \phi\}^M)$$

$$(\Delta\bar{\chi}_-, \{\bar{\phi}, \bar{\phi}\}^{\bar{M}}) = -2(\bar{\phi}, \{\Delta\bar{\phi}, \bar{\chi}_-\}^{\bar{N}}) = -2(\bar{\chi}_-, \{\Delta\bar{\phi}, \bar{\phi}\}^{\bar{M}})$$

That is the Leibniz rule on lattice. $(\Delta = -\Delta^T)$
The rule with locality is forbidden by the No-go theorem.

Restriction for SUSY

Instead, we restrict for $N = 2$ SUSY $\delta_{\epsilon,\bar{\epsilon}} \to \delta_{\epsilon}$
($\epsilon_{\pm}$ are omitted.)

$$
\begin{aligned}
\delta_+\phi &= \chi_- & \qquad \delta_-\phi &= 0 \\
\delta_+\bar{\phi} &= 0 & \qquad \delta_-\bar{\phi} &= \bar{\chi}_+ \\
\delta_+\chi_+ &= F & \qquad \delta_-\chi_+ &= i\Delta\phi \\
\delta_+\bar{\chi}_- &= i\Delta\bar{\phi} & \qquad \delta_-\bar{\chi}_- &= \bar{F} \\
\delta_+\bar{\chi}_+ &= 0 & \qquad \delta_-\bar{\chi}_+ &= 0 \\
\delta_+\chi_- &= 0 & \qquad \delta_-\chi_- &= 0 \\
\delta_+F &= 0 & \qquad \delta_-F &= -i\Delta\chi_- \\
\delta_+\bar{F} &= -i\Delta\bar{\chi}_+ & \qquad \delta_-\bar{F} &= 0
\end{aligned}
$$

Invariance of restricted (half of) SUSY

Restriction for SUSY $\delta_{\epsilon,\bar{\epsilon}} \to \delta_{\epsilon}$

$$S = S_0(+S_{ds}) + S_m + S_{int}$$

$$\delta_{\epsilon}\Big(S_0, (S_{ds}), S_m, S_{int}\Big) = 0$$

S_{ds} is a doubler-suppressing term like Wilson one.
$\to S_{ds} = \delta_{\epsilon}(\phi, H\psi)$, $\hat{H}(w) = r/2(1 - w - 1/w)$

Each term is invariant when $\Delta = -\Delta^T$ and

$$(\Delta\chi_-, \{\phi, \phi\}^M) = -2(\phi, \{\chi_-, \Delta\phi\}^N) = -(\Delta\phi, \{\phi, \chi_-\}^M) - (\Delta\phi, \{\chi_-, \phi\}^M)$$

$$(\Delta\bar{\chi}_-, \{\bar{\phi}, \bar{\phi}\}^{\bar{M}}) = -2(\bar{\phi}, \{\bar{\chi}_-, \Delta\bar{\phi}\}^{\bar{N}}) = -(\Delta\bar{\phi}, \{\bar{\phi}, \bar{\chi}_-\}^{\bar{M}}) - (\Delta\bar{\phi}, \{\bar{\chi}_-, \bar{\phi}\}^{\bar{M}})$$

CLR- holomorphic functional equations

Cyclic Leibniz rule(CLR)
(M and $\bar{M}$ in $\{,\}^M$ and $\{,\}^{\bar{M}}$ are omitted.) $\quad \Delta^T = -\Delta$,

$$(\Delta\phi, \{\psi, \chi\}) + (\Delta\psi, \{\chi, \phi\}) + (\Delta\chi, \{\phi, \psi\}) = 0$$

$$\uparrow \;\; \mathrm{LR}: \quad (\Delta\phi, \{\psi, \chi\}) + (\phi, \{\Delta\psi, \chi\}) + (\phi, \{\psi, \Delta\chi\}) = 0, \quad \leftarrow \times$$

$$(A, \{B, C\}) = (B, \{C, A\}) \;\; \leftarrow \times$$

Expression by a holomorphic function

$$P(w,z) + P(z, 1/(wz)) + P(1/(wz), w) = 0$$

$$P(w,z) \equiv \hat{\Delta}(1/(wz))\hat{M}(w,z)$$

$$P(w,z) = P(z,w), \;\; P(w, 1/w) = 0$$

$P(w,z) \leftrightarrow$ a holomorphic set of $\hat{\Delta}(w)$ and $\hat{M}(w,z)$

Simple Local Product

Warning! A simple ordinary local product

$$M_{k\ell m} = \delta_{k\ell}\delta_{km} \leftrightarrow \hat{M}(w,z) = 1$$

is a non-holomorphic set of CLR. If we take it, CLR

$$P(w,z) \equiv \hat{\Delta}(1/(wz))\hat{M}(w,z)$$

$$P(w,z) + P(z,1/(wz)) + P(1/(wz),w) = 0$$

leads us to non-holomorphic $\hat{\Delta}$:

$$\hat{\Delta}(1/(wz)) + \hat{\Delta}(w) + \hat{\Delta}(z) = 0$$

$$\rightarrow \hat{\Delta}(w) \sim \mathrm{Log}\, w, \;\; w = \exp(iap)$$

$\therefore \Delta$ is nonlocal (SLAC-type). Actually,

$$(A, \{B, C\}^M) = (B, \{C, A\}^M) \;\; \rightarrow \text{hold}$$

General Solutions of CLR

We found general solution by any holomorphic function

$$
\begin{aligned}
P(w,z) \quad &= f(w,z) + f(z,1) + f(wz, 1/(wz)) \\
&+ f(z,w) + f(w,1) + f(wz, 1/(wz)) \\
&- f(z, 1/(wz)) - f(1/(wz), 1) - f(1/w, w) \\
&- f(w, 1/(wz)) - f(1/(wz), 1) - f(1/z, z)
\end{aligned}
$$

where $f(w,z)$ is an arbitrary holomorphic function.

· A simple example. Take $f(w,z) = (w^2 + z^2)/12$.

$$P(w,z) = \frac{w^2 - w^{-2} + z^2 - z^{-2} + 2(wz)^2 - 2(wz)^{-2}}{12},$$

Remind $P(w,z) = \hat{\Delta}(1/(wz))\hat{M}(w,z)$.

$$\hat{\Delta}(w) = \frac{w - w^{-1}}{2},\ \hat{M}(w,z) = \frac{1}{6}(2(wz + (wz)^{-1}) + wz^{-1} + w^{-1}z)$$

CLR realization

In a real lattice space, this example implies

$$\{\phi, \chi\}_n =$$

$$\frac{1}{6}(2\phi_{n+1}\chi_{n+1} + 2\phi_{n-1}\chi_{n-1} + \phi_{n+1}\chi_{n-1} + \phi_{n-1}\chi_{n+1}),$$

$$(\Delta\phi)_n = \frac{\phi_{n+1} - \phi_{n-1}}{2}.$$

Locally realized!

CLR $\cdots$ our key relation

$$(\Delta\phi, \{\psi, \chi\}) + (\Delta\psi, \{\chi, \phi\}) + (\Delta\chi, \{\phi, \psi\}) = 0.$$

This is the origin of cyclic Leibniz rule's naming .

Consequently,

$$S = S_0 + S_m + S_{ds} + S_{int}$$

$$= (\Delta\phi, \Delta\bar{\phi}) + i(\bar{\chi}_+, \Delta\chi_+) + i(\bar{\chi}_-, \Delta\chi_)) + (\bar{F}, F) + im(F, \phi) + im(\chi_+, \chi_-)$$

$$+ i(F, H\phi) + i(\bar{\chi}_+, H\chi_-) + ig_2(F, \{\phi, \phi\}^M) - 2ig_2(\chi_+, \{\chi_-, \phi\}^M) + \text{c.c}$$

$$\rightarrow \delta_\pm S = 0$$

To combine Δ with $\{,\}^M \rightarrow$ CLR.
We remark on the number of exact supersymmetry:

ϵ_+ and ϵ_-

Two SUSY charges are exactly conserved!

Nicolai mapping and other approaches

Our lattice model has two Nicolai mappings.
For simplicity, we consider $W_n(\phi) = g_2\{\phi, \phi\}_n$.

The mapping by other approach

M. Beccaria et al. PRD58(1998)065009
S. Catterall and E. Gregory, PLB487(2000)349

Their models cannot have two or more Nicolai mappings.
Because they have a surface term problem.

Our model is controlled under CLR and
free from the surface term problem.

Nicolai mappings of CSQM

Two Nicolai mappings in our model

$$\xi_n^{\pm} = (\Delta\bar{\phi})_n \pm i((m\phi)_n + (H\phi)_n + g_2\{\phi,\phi\}_n^{M}),$$

$$\bar{\xi}_n^{\pm} = (\Delta\phi)_n \pm i((\bar{m}\bar{\phi})_n + (\bar{H}\bar{\phi})_n + g_2^*\{\bar{\phi},\bar{\phi}\}_n^{\bar{M}}).$$

Surface terms for $(\bar{\xi}^{\pm}, \xi^{\pm})$

$$(\Delta\phi,\phi) = (\Delta\phi, H\phi) = (\Delta\bar{\phi},\bar{\phi}) = (\Delta\bar{\phi},\bar{H}\bar{\phi}) = 0,$$

$$(\Delta\phi,\{\phi,\phi\}^{M}) = (\Delta\bar{\phi},\{\bar{\phi},\bar{\phi}\}^{\bar{M}}) = 0$$

are exactly vanished. (The latter eqs. are exactly two CLRs.)
The mapping by other approach is one even in SCQM.
$\rightarrow$ only an exact supercharge is keeping in other appraoch.

Symmetry of CSQM

Symmetry of CSQM
Four supercharges in the (continuum) theory

$$\delta_\pm, \ \times\ \bar{\delta}_\pm$$

For our formulation, two charges are conserved.

$$\delta_\pm^2 = 0,\ \{\delta_+, \delta_-\} = 0$$
$$S = S_0 + S_1 = \delta_+\delta_-\mathcal{O}_0 + \delta_+\mathcal{O}_+ + \delta_-\mathcal{O}_-$$

The kinetic term

$$S_0 = \delta_+\delta_-(\chi_+, \bar{\chi}_-)$$

Interaction terms (+ mass terms and doubler-suppressed terms)

$$S_1 = S_{\rm int} + S_{\rm mass} + S_{\rm ds} = \delta_+\mathcal{O}_+ + \delta_-\mathcal{O}_-$$
$$S_{\rm int} = -ig_2\delta_+(\chi_+, \{\phi, \phi\}^M) + ig_2^*\delta_-(\bar{\chi}_-, \{\bar{\phi}, \bar{\phi}\}^{\bar{M}}) + \cdots$$

Summary

With interactions in our formlulation, it is possible

- to establish CLR
  $(\Delta\phi, \{\psi, \chi\}) + (\Delta\psi, \{\chi, \phi\}) + (\Delta\chi, \{\phi, \psi\}) = 0$
  with locality and translational invariance
- to construct quantum mechanical local models on lattice with exact supersymmetry
- to make superspace formulation
- to find a Nicolai map without surface terms
- to calculate its Witten index $\leftarrow$ localization technique
- to extend to $N-$body product
  $\rightarrow (\Delta\phi, \{\phi, \phi, \cdots, \phi\}) = 0$

Discussion

- more conserved supercharges in the complex model($N=2$) are realized.
  $\leftrightarrow$ more Nicolai mappings than other approach are found.
- SUSY algebra
  Although our realization is still

$$\{Q, Q\} = 0,$$

  $\delta_{\pm}$ includes Δ .
  $\rightarrow$ Strong constraint for effective action, effective potential!
  $\Rightarrow$ exact mass degeneracy bet. boson and fermion
  $\Rightarrow$ CLR is kept under any quantum fluctuation!