Resonances and Multi-Particle States

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HISKP

bctp Bethe Center for Theoretical Physics

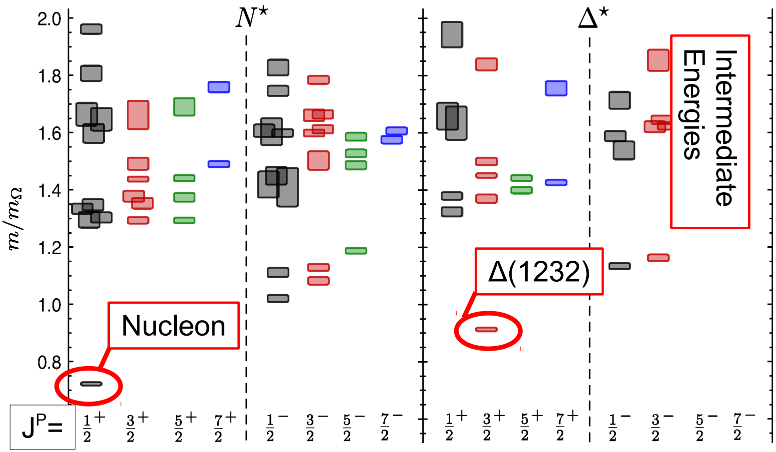
LATTICE 13

JULY 29 – AUGUST 03 2013
MAINZ, GERMANY

August 3, 2013

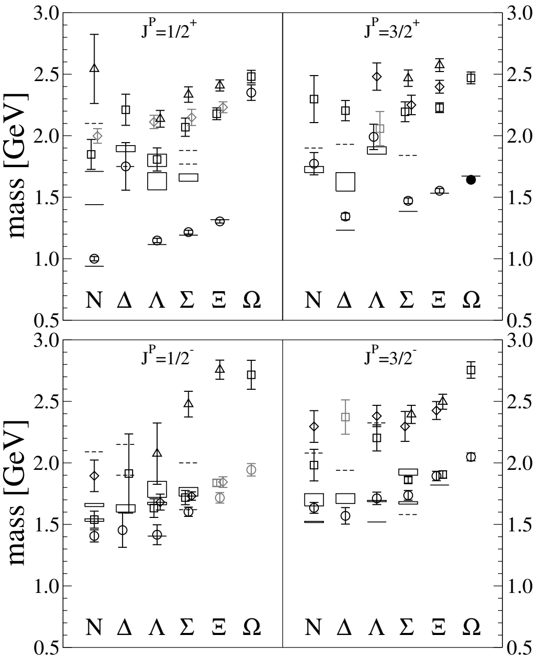
The baryon spectrum: N^* and Δ resonances

- Many resonances predicted in lattice calculations — Missing resonance problem from quark model reappears [Edwards *et al.*, Phys.Rev. D84 (2011)]:



$m_\pi = 396 MeV$ (!)

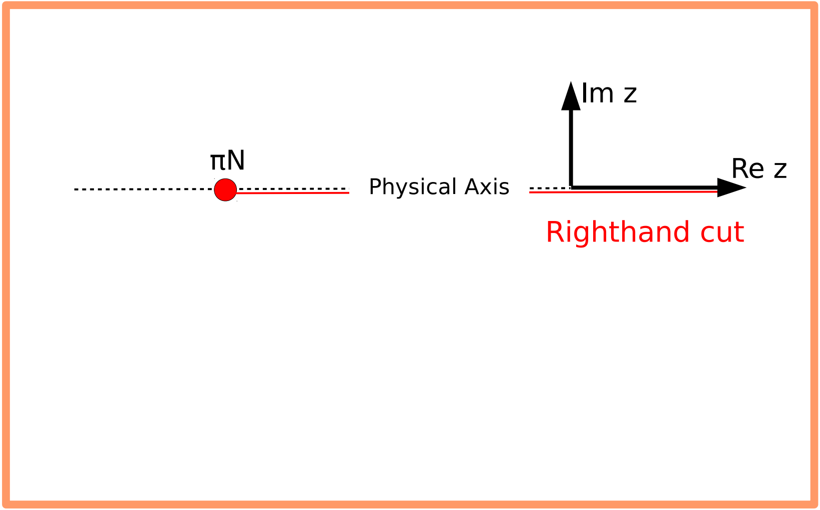
New results from baryon spectroscopy



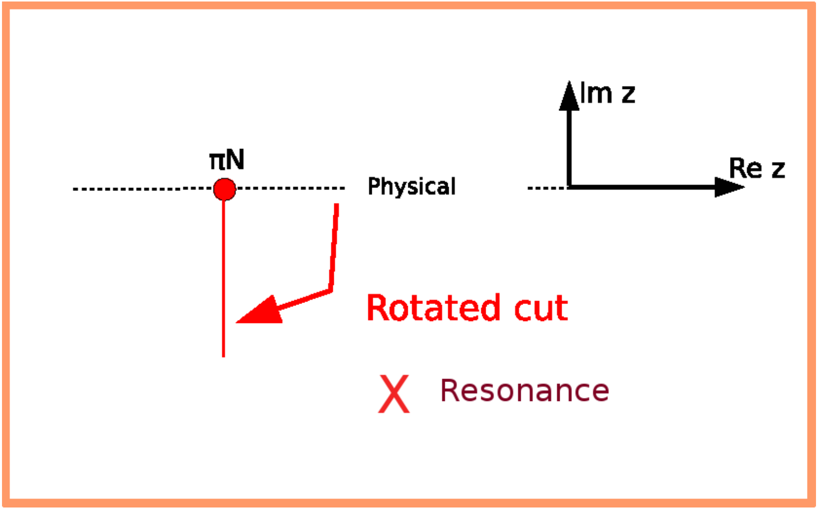
- Seven ensembles
- 200-300 gauge configurations each
- M_π from 255 to 596 MeV
- Lattice size $16^3 \times 32$, $12^3 \times 24$, $24^3 \times 48$
- Lattice spacing between 0.1324 and 0.1398 fm
- Extrapolated to the physical pion mass; shown plot: not yet finite-volume corrected

[Engel, Lang, Mohler, Schäfer, PRD (2013)]

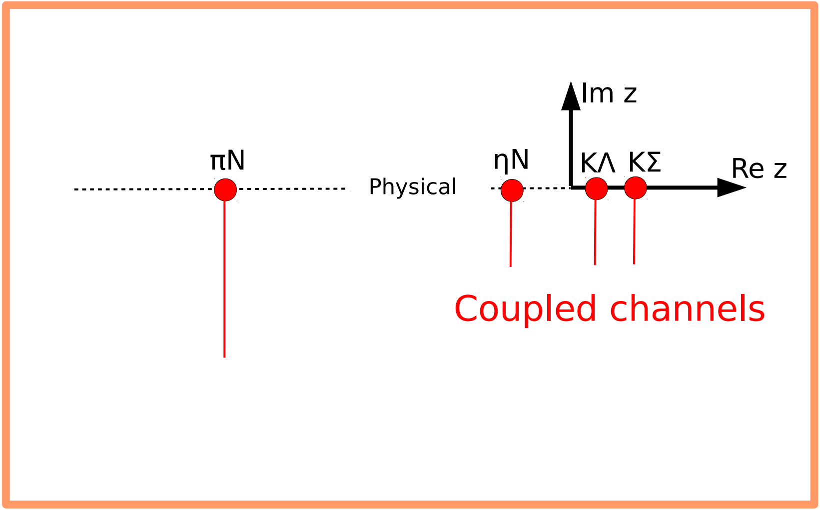
$(z = E)$



$(z = E)$



$(z = E)$



Fit to meson-meson data using unitary ChPT with NLO terms

[M.D., Meißner, JHEP (2012)] using IAM [Oller, Oset, Peláez, PRC (1999)]

Unitary extension of ChPT, can be matched to ChPT order-by-order.

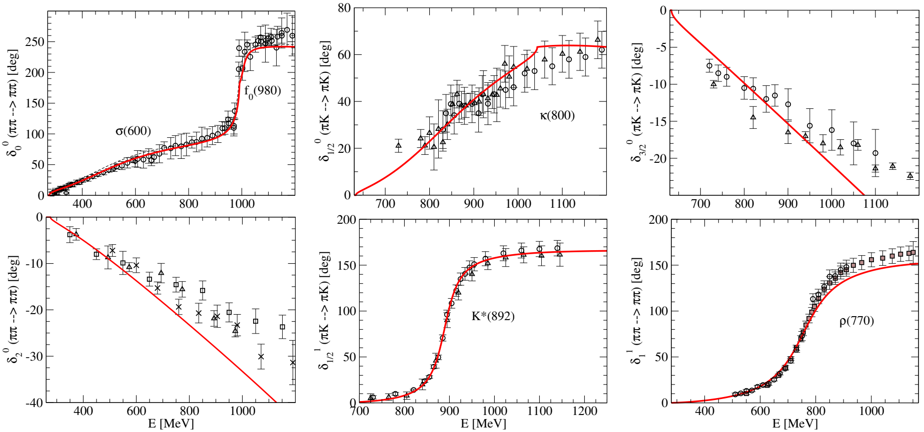
Table: Fitted values for the L_i $[\times 10^{-3}]$ and $q_{\max}$ [MeV].

L_1	L_2	L_3	L_4
$0.873^{+0.017}_{-0.028}$	$0.627^{+0.028}_{-0.014}$	-3.5 [fixed]	$-0.710^{+0.022}_{-0.026}$
L_5	$L_6 + L_8$	L_7	$q_{\max}$ [MeV]
$2.937^{+0.048}_{-0.094}$	$1.386^{+0.026}_{-0.050}$	$0.749^{+0.106}_{-0.074}$	981 [fixed]

- A resonance is characterized by its (complex) pole position and residues, corresponding to resonance mass, width, and branching ratio.

Table: Pole positions z_0 [MeV] and residues $a_{-1}[M_\pi]$ in different channels.
I, L, S: isospin, angular momentum, strangeness.

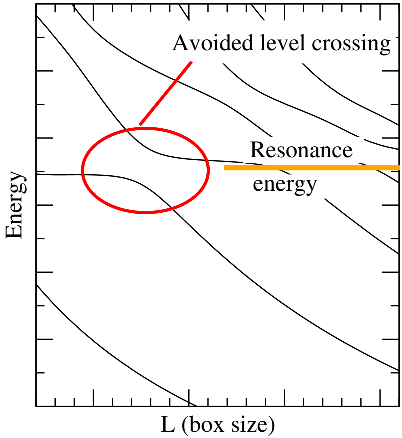
I	L	S	Resonance	sheet	z_0 [MeV]	a_{-1} $[M_\pi]$	a_{-1} $[M_\pi]$
0	0	0	$\sigma(600)$	pu	$434+i\,261$	$-31-i\,19\,(\bar{K}K)$	$-30+i\,86\,(\pi\pi)$
0	0	0	$f_0(980)$	pu	$1003+i\,15$	$16-i\,79\,(\bar{K}K)$	$-12+i\,4\,(\pi\pi)$
1/2	0	-1	$\kappa(800)$	pu	$815+i\,226$	$-36+i\,39\,(\eta K)$	$-30+i\,57\,(\pi K)$
1	0	0	$a_0(980)$	pu	$1019-i\,4$	$-10-i\,107\,(\bar{K}K)$	$21-i\,31\,(\pi\eta)$
0	1	0	$\phi(1020)$	p	$976+i\,0$	$-2+i\,0\,(\bar{K}K)$	—
1/2	1	-1	$K^*(892)$	pu	$889+i\,25$	$-10+i\,0.1\,(\eta K)$	$14+i\,4\,(\pi K)$
1	1	0	$\rho(770)$	pu	$755+i\,95$	$-11+i\,2\,(\bar{K}K)$	$33+i\,17\,(\pi\pi)$



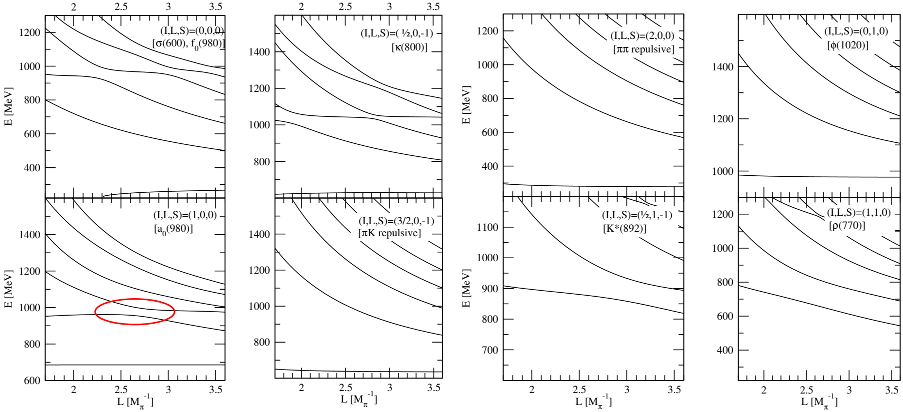
δ_0^0 (ππ --> ππ) [deg]
σ(600)
f_0(980)
$\delta_{1/2}^0$ (πK --> πK) [deg]
κ(800)
$\delta_{3/2}^0$ (πK --> πK) [deg]
δ_2^0 (ππ --> ππ) [deg]
$\delta_{1/2}^1$ (πK --> πK) [deg]
K*(892)
δ_1^1 (ππ --> ππ) [deg]
ρ(770)
E [MeV]
E [MeV]
E [MeV]

Resonances decaying on the lattice

Eigenvalues in the finite volume



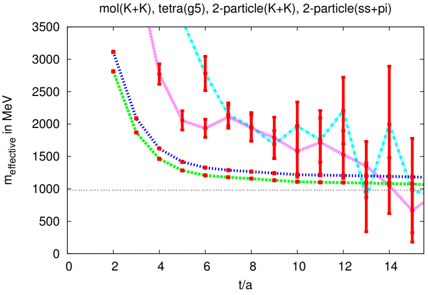
Prediction of levels (also for $M_\pi \neq M_\pi^{\text{phys.}}$)

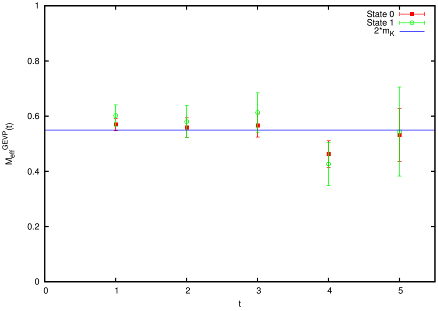


[M.D., Meißner, JHEP (2012)
Loops in t- and u-channel (1-loop calculation): [Albaladejo, Rios, Oller, Roca, arXiv: 1307.5169; Albaladejo, Oller, Oset, Rios, Roca, JHEP (2013)]

The $a_0(980)$

[Wagner, Daldrop, Abdel-Rehim, Urbach *et. al.* [ETMC], JHEP (2013) & new results]



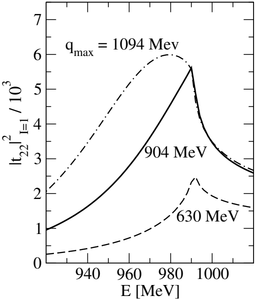


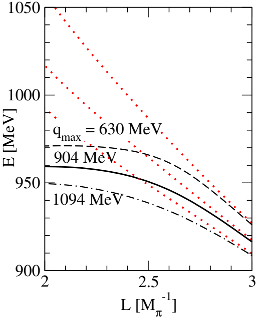
- $M_\pi \sim 300$ MeV, no singly disconnected diagrams.
- Operators: $\bar{K}K$ molecular, diquark-antidiquark, meson-meson.
- Two low-lying states, large overlap with meson-meson.

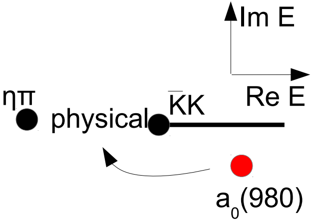
- $M_\pi \sim 300$ MeV, singly disconnected diagrams included.
- Operators: $q\bar{q}$, $\bar{K}K$ molecular.
- Again, two low lying states, no information on additional state.

The $a_0(980)$ in a multi-channel environment

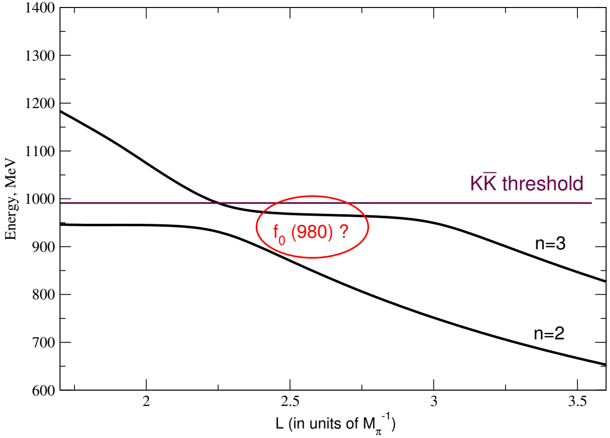
M.D., Meißner, Oset, Rusetsky, EPJA (2011); see also Lage, Meißner, Rusetsky, PLB (2009)







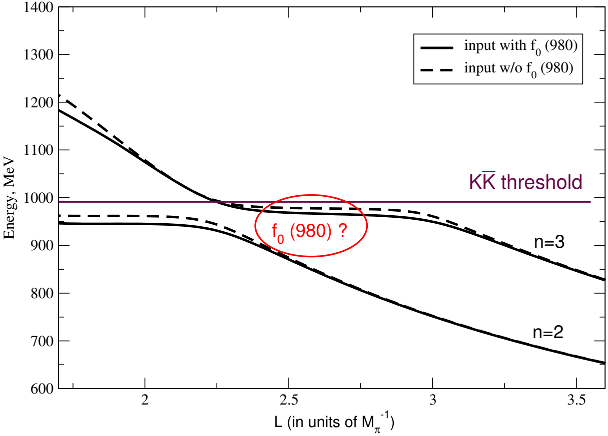
Avoided level crossing in the energy levels



Using Unitarized Chiral Perturbation Theory to produce energy levels

[M. D., Meißner, Oset, Rusetsky, EPJA (2011)]

Changing the input . . .



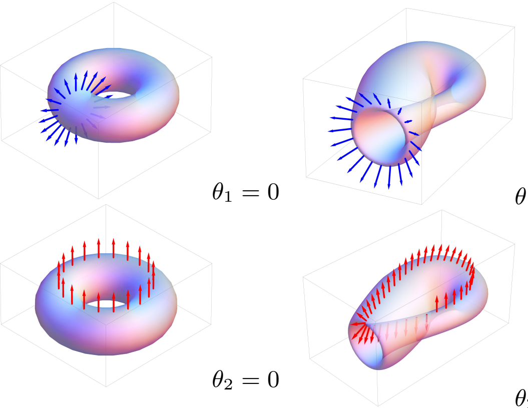
- Weaker coupling to the $K\bar{K}$ channel, $f_0(980)$ disappears
- Avoided level crossing still occurs at the same place: threshold!
- The threshold can be moved by using twisted b.c.!

Twisting the boundary conditions (B.C.)

[Bernard, Lage, Meißner, Rusetsky, JHEP (2011), M.D., Meißner, Oset, Rusetsky, EPJA (2011)]

- Periodic B.C.:
  $\Psi(\vec{x}+\hat{\mathbf{e}}_i L) = \Psi(\vec{x})$
- Periodic in 2 dim.:

- Twisted B.C.:
  $\Psi(\vec{x}+\hat{\mathbf{e}}_i L) = e^{i\theta_i}\Psi(\vec{x})$
- Periodic/antiperiodic:

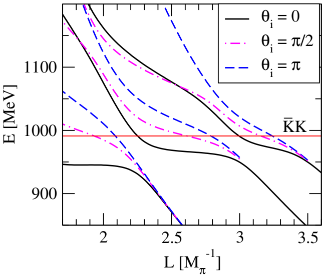


$\theta_1 = 0$ $\theta_2 = 0$ (periodic)

$\theta_1 = 0$ $\theta_2 = \pi$ (periodic/antiperiodic)

Example: the $f_0(980)$

- S-wave, coupled-channels $\pi\pi$, $\bar{K}K$.
- Twisted B.C. for the s-quark:
  $u(\vec{x}+\hat{\mathbf{e}}_i L) = u(\vec{x})$
  $d(\vec{x}+\hat{\mathbf{e}}_i L) = d(\vec{x})$
  $s(\vec{x}+\hat{\mathbf{e}}_i L) = e^{i\theta_i}s(\vec{x})$
- Three unknown transitions
  - $V(\pi\pi \to \pi\pi)$
  - $V(\pi\pi \to \bar{K}K)$
  - $V(\bar{K}K \to \bar{K}K)$



Good news: Partial twisting = full twisting in some situations [A. Rusetsky, Lattice 13]

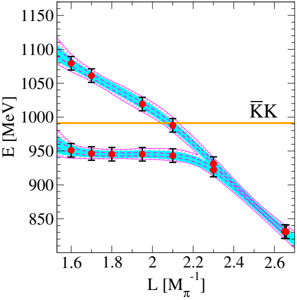
Extracting resonances from lattice data
[M.D., Meißner, Oset/Rusetsky, EPJA **47** (2011)]

- Need for an interpolation in energy ($\rightarrow$ Unitarized ChPT)
- Expand a two-channel transition V in energy (i, j: $\pi\pi$, $\bar{K}K$):

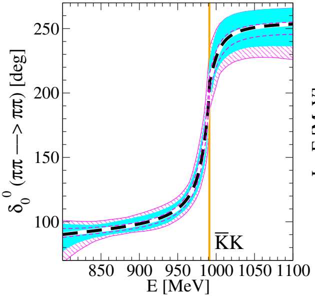
$$V_{ij}(E) = a_{ij} + b_{ij}(E^2 - 4M_K^2)$$

- Include model-independently known LO contribution in a, b.
- Or even NLO contributions (more fit parameters) [M.D., Meißner, JHEP (2012)].

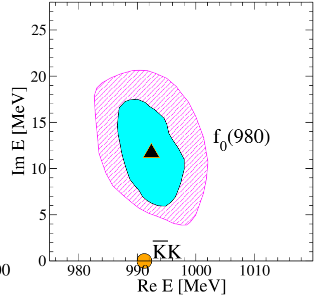
lattice data & fit



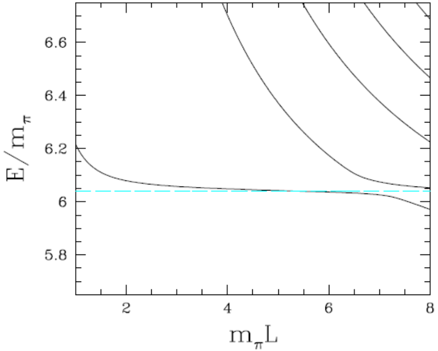
extracted phase shift



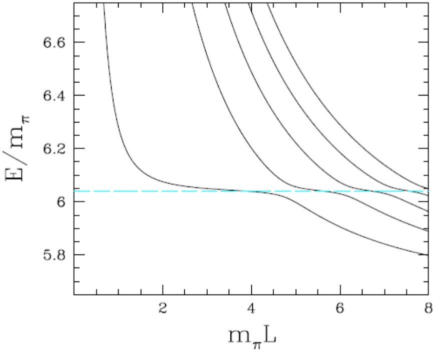
$f_0(980)$ pole position



Moving frames to scan narrow resonances ($\Sigma^* \to \pi\Lambda$)

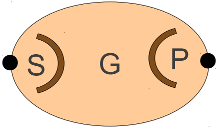


$\boldsymbol{P} = 0$



$\boldsymbol{P} = \frac{2\pi}{L}(0, 0, 1)$

- Infinite volume limit: Rotational symmetry



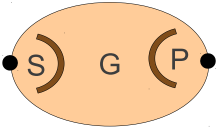
$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}|)\; Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim \delta_{\ell\ell'}\delta_{mm'}\,.$$

S→S	0	0	0
0	$P_{-1} \to P_{-1}$	0	0
0	0	$P_0 \to P_0$	0
0	0	0	$P_1 \to P_1$

Mixing of partial waves

Example: S- and P-waves

- Infinite volume limit: Rotational symmetry



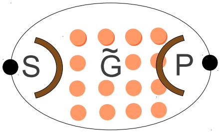
$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}|)\; Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim \delta_{\ell\ell'}\delta_{mm'} \,.$$

- Wigner-Eckart theorem:

S→S	0	0	0
0	P_{-1} →	0	0
0	0		0
0	0	0	→ P_1

Equal

- Finite volume: Rotational symmetry $\to$ Cubic symmetry



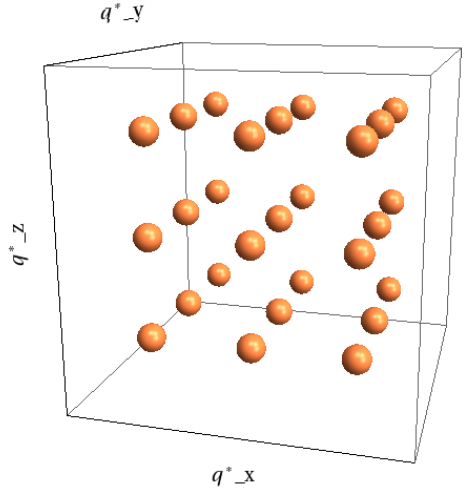
$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' m m'}\,.$$

- $S-G$-wave mixing, but $S-P$ waves still orthogonal:

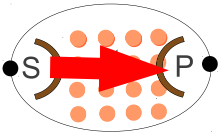
S→S	0	0	0
0	$P_{-1} \to P_{-1}$	0	0
0	0	$P_0 \to P_0$	0
0	0	0	$P_1 \to P_1$

Breaking of cubic symmetry through boost

Example: Lattice points $\vec{q}^*$ boosted with $P = (0,0,0) \to \frac{2\pi}{L}(0,0,2)$:



- Finite volume & boost: Cubic symmetry $\to$ subgroups of cubic symmetry

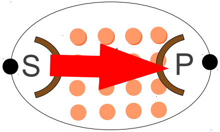


$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' m m'}\,.$$

- For boost $P = \frac{2\pi}{L}$ (0,1,1):

$S \to S$	0	$S \to P_0$	0
0	$P_{-1} \to P_{-1}$	0	$P_{-1} \to P_1$
$P_0 \to S$	0	$P_0 \to P_0$	0
0	$P_1 \to P_{-1}$	0	$P_1 \to P_1$

- Finite volume & boost: Cubic symmetry → subgroups of cubic symmetry



$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' m m'} .$$

- More complicated boosts:

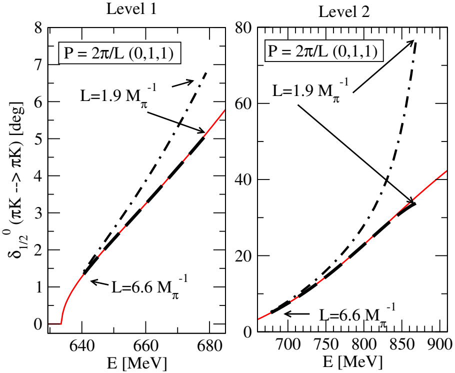
S→S	S→P_{-1}	S→P_0	S→P_1
P_{-1} → S	P_{-1} → P_{-1}	P_{-1} → P_0	P_{-1} → P_1
P_0→S	P_0 → P_{-1}	P_0 → P_0	P_0 → P_1
P_1 → S	P_1 → P_{-1}	P_1 → P_0	P_1 → P_1

Disentanglement of partial waves [M.D., Meißner, Oset, Rusetsky, EPJA (2012)]

Example: S- and P-waves for the $\kappa(800)/K^*(892)$ system

Knowledge of P-wave (from separate analysis of lattice data) allows to disentangle the S-wave:

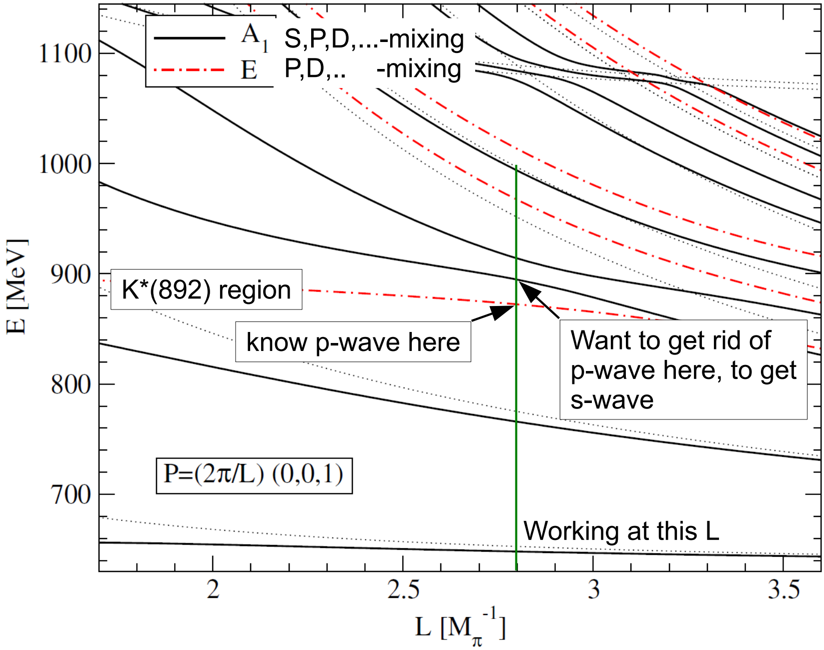
$$p\,\cot\delta_S = -8\pi\,E\,\frac{p\,\cot\delta_P\,\hat{G}_{SS} - 8\pi E(\hat{G}_{SP}^2 - \hat{G}_{SS}\hat{G}_{PP})}{p\,\cot\delta_P + 8\pi\,E\,\hat{G}_{PP}}$$



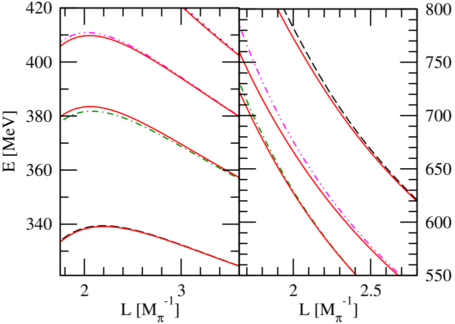
- $\delta_S \equiv \delta^0_{1/2}(\pi K \to \pi K)$
- Red solid: Actual S-wave phase shift.
- Dash-dotted: Reconstructed S-wave phase shift, PW-mixing ignored.
- Dashed: Reconstructed S-wave phase shift, PW-mixing disentangled.
- small p-wave: Level shift $\Delta E \simeq -\dfrac{6\pi E_S \delta_P}{L^3 p\omega_1\omega_2}$

Rummukainen, Gottlieb, NPB (1995); Kim, Sachrajda, Sharpe, NPB (2005); Davoudi, Savage, PRD (2011), Z. Fu, PRD (2012); Leskovec, Prelovsek, PRD (2012); Dudek, Edwards, Thomas, PRD (2012); Hansen, Sharpe, PRD 86 (2012); Briceño, Davoudi, arXiv:1204.1110; Göckeler, Horsley, Lage, Meißner, Rakow, Rusetsky, Schierholz, Zanotti, PRD (2012)

The need for an interpolation in energy ($K\pi$ scattering)



Mixing of partial waves in boosted multiple channels: $\sigma(600)$
[M.D., E. Oset, A. Rusetsky, EPJA (2012)]



Solid: Levels from A_1^+.
Non-solid: Neglecting the D-wave.

- $\pi\pi$ & $\bar{K}K$ in S-wave, $\pi\pi$ in D-wave.
- Organization in Matrices (A_1^+), e.g. $\vec{P} = (2\pi/L)(0,0,1)$, $(2\pi/L)(1,1,1)$, and $(2\pi/L)(0,0,2)$:

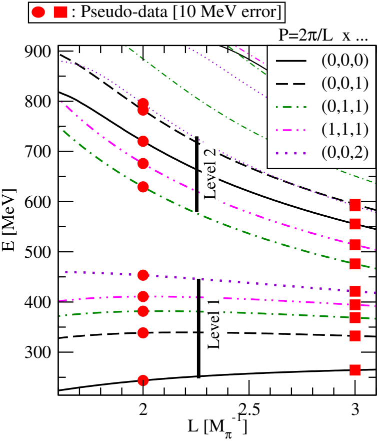
$$V = \begin{pmatrix} V_S^{(11)} & V_S^{(12)} & 0 \\ V_S^{(21)} & V_S^{(22)} & 0 \\ 0 & 0 & V_D^{(22)} \end{pmatrix}$$

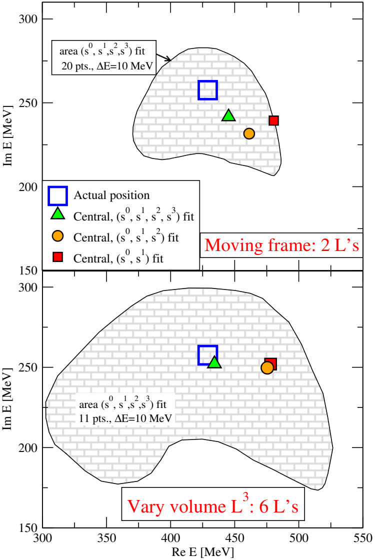
$$\tilde{G} = \begin{pmatrix} \tilde{G}^{R\,(1)}_{00,00} & 0 & 0 \\ 0 & \tilde{G}^{R\,(2)}_{00,00} & \tilde{G}^{R\,(2)}_{00,20} \\ 0 & \tilde{G}^{R\,(2)}_{20,00} & \tilde{G}^{R\,(2)}_{20,20} \end{pmatrix}$$

- Phase extraction (κ): Expand and fit V_S, V_P *simultaneously* to different representations instead of
  1. P-wave from B_1, B_2, E
  2. S-wave from P and A_1

  (reduction of error).

- The first two levels for the first five boosts:



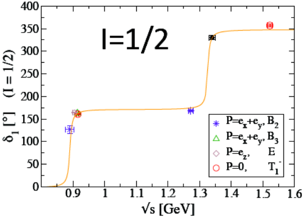


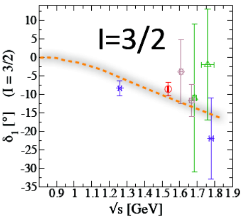
Kπ scattering and K* width in moving frames

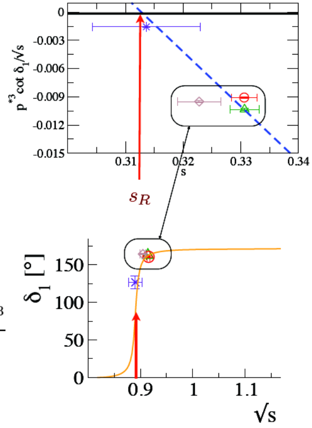
Prelovsek, Leskovec, Lang, Mohler, this conf. and *arXiv: 1307.0736*

p-wave, coupled system of 5 $q\bar{q}$ and 3 Kπ operators, total momentum P=(000),(001),(011)

Representations B_2, B_3, E, T_1^-





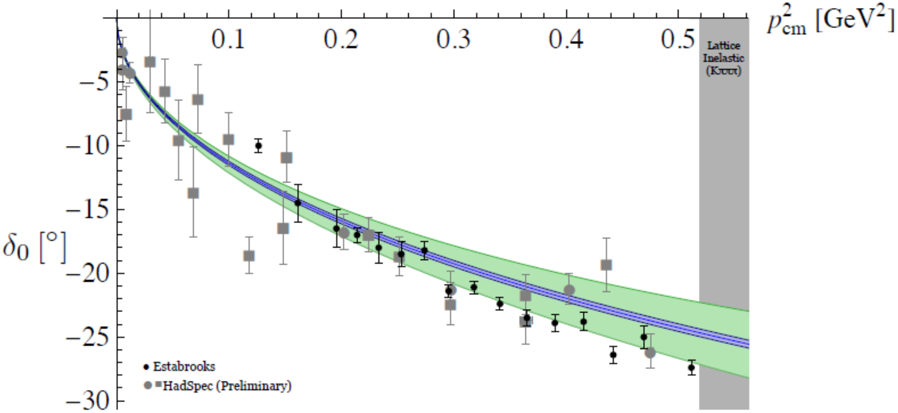


$$\frac{p^{*3}}{\sqrt{s}}\cot\delta_1(s) = \left[\sum_{K^*} \frac{g^2_{K^*_i}}{6\pi}\,\frac{1}{m^2_{K^*_i} - s}\right]^{-1} \qquad \Gamma[K^* \to K\pi] = \frac{g^2}{6\pi}\,\frac{p^{*3}}{s}$$

	$m_{K^*(892)}$ [MeV]	$g_{K^*(892)}$ [no unit]	$m_{K^*(1410)}$ [GeV]	$g_{K^*(1410)}$ [no unit]
lat	891 ± 14	5.7 ± 1.6	1.33 ± 0.02	input
exp	891.66 ± 0.26	5.72 ± 0.06	1.414 ± 0.0015	1.59 ± 0.03

$K\pi$ scattering in $I = 3/2$

D. Wilson [HadSpec], this conf.



Nπ (1/2⁻) channel

C.B. Lang and V. Verduci, this conf. and Phys.Rev. D 87, 054502 (2013)

m_π=266 MeV; distillation method; variational analysis using a basis of N (3 quarks) and Nπ (5 quarks) interpolators;

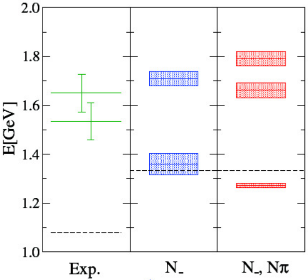
$$(N_{\pm}^{(i)})_\mu(\vec{p}=0) = \sum_{\vec{x}} \epsilon_{abc} \left(P_\pm \Gamma_1^{(i)} u_a(\vec{x})\right)_\mu \left(u_b^T(\vec{x}) \Gamma_2^{(i)} d_c(\vec{x})\right)$$

$$\pi^+(\vec{p}=0) = \sum_{\vec{x}} \overline{d}_a(\vec{x})\gamma_5 u_a(\vec{x}) \, ,$$

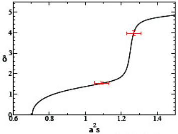
$$\pi^0(\vec{p}=0) = \sum_{\vec{x}} \frac{1}{\sqrt{2}} \left(\overline{u}_a(\vec{x})\gamma_5 u_a(\vec{x}) - \overline{d}_a(\vec{x})\gamma_5 d_a(\vec{x})\right)$$

$$O_{N\pi}(I=\frac{1}{2}, I_3=\frac{1}{2}) = p\pi^0 + \sqrt{2}\, n\pi^+$$

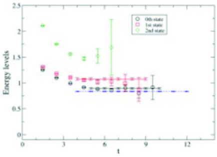
$$N\pi(\vec{p}=0) = \gamma_5 N_+(\vec{p}=0)\pi(\vec{p}=0)$$



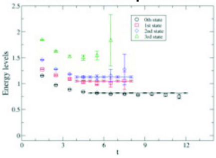
Lüscher relation
➛ phase shift:



only udd operators:



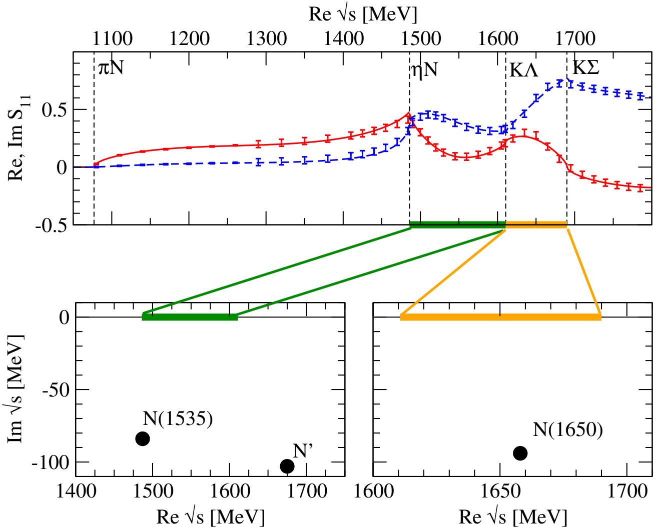
udd and uddud operators:



Assuming 2 elastic resonances with identical coupling we get m_1=1.678 GeV

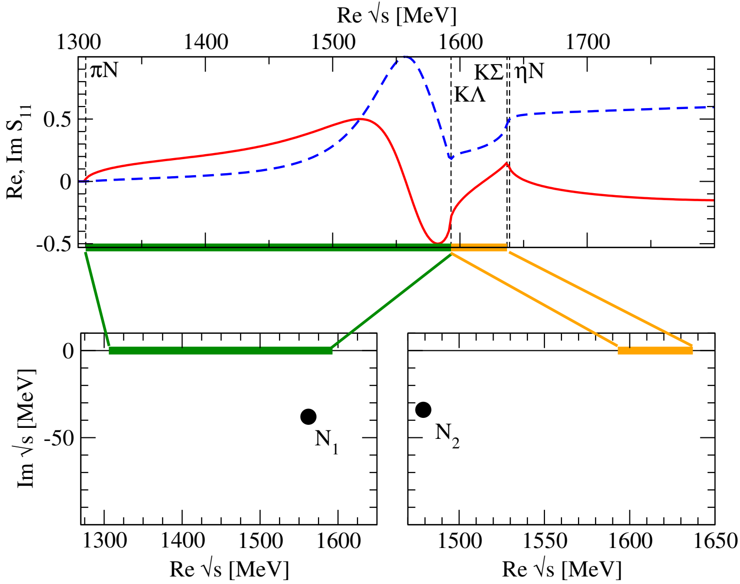
m_2=1.873 GeV

- Unitarized chiral interaction with NLO contact terms

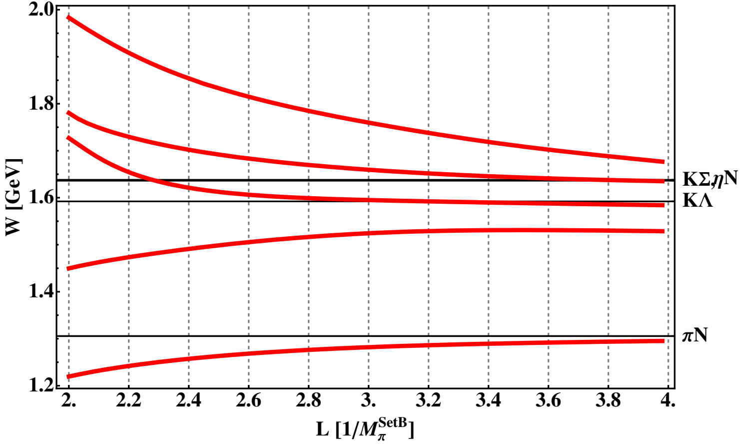


Data: SAID (2006)

Chiral extrapolation to a QCDSF lattice setup

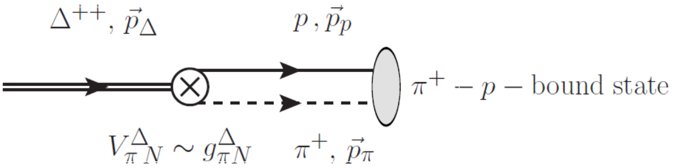


Prediction of the lattice spectrum

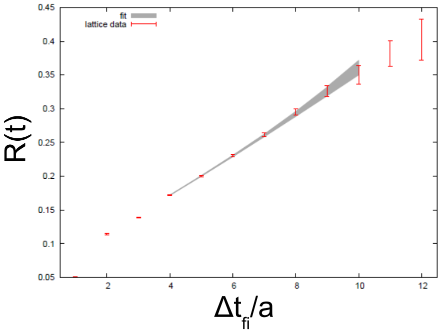


- No one-to-one mapping of levels to resonances → coupled channel analysis; hidden poles appear.

Δ resonance from the transfer matrix method [ETMC, arXiv:1305.6081]

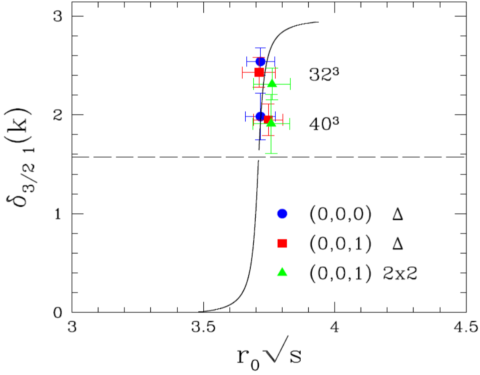


$$R(\Delta t_{fi}, \vec{Q}, \vec{q}) = \frac{C_\mu^{\Delta\to\pi N}(\Delta t_{fi}, \vec{Q}, \vec{q})}{\sqrt{C_\mu^{\Delta}(\Delta t_{fi}, \vec{Q})\, C^{\pi N}(\Delta t_{fi}, \vec{Q}, \vec{q})}} \sim \langle \Delta | H | \pi N \rangle \Delta t_{fi}$$



$$\langle \Delta | H | \pi N \rangle \rightarrow g^{\Delta}_{\pi N}$$
$$g^{\Delta}_{\pi N}(\mathrm{lat}) = 27.0 \pm 0.6 \pm 1.5$$

Rakow/QCDSF [this conference]:
$\pi^+ p \rightarrow K^+\Sigma^+$: σ~0.4 (0.2) fm
at SU(3) symmetrical point



Effective range formula

$$\frac{k^3}{E}\cot\,\delta_{3/2\,1}(k) = \frac{24\pi}{g^2_{\Delta N\pi}}\left(m^2_\Delta - E^2\right)$$

$$\Gamma_\Delta = \frac{g^2_{\Delta N\pi}}{24\pi}\frac{k^3_\Delta}{m^2_\Delta} \qquad \frac{g^2_{\Delta N\pi}}{4\pi} = 14.6$$

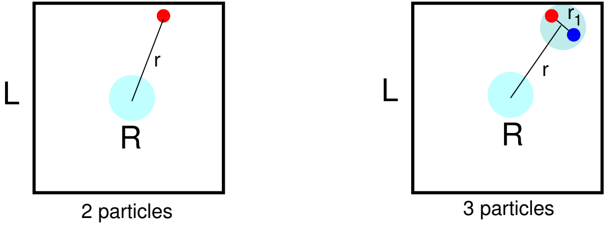
- $\Lambda(1405)$: Eigenlevels found, one of them below $\bar{K}N$ threshold.
- Suppression of the strange quark electromagnetic form factor $\rightarrow$ large $\bar{K}N$ component.

[Menadue, Kamleh, Leinweber, Mahbub [CSSM], PRL (2012); Menadue, this conference]

Three particles:

New finite volume methods.

Three particles in a finite volume



- In case of 2 particles: $r \gg R$, when particles are near the walls
- In case of 3 particles: it may happen that $r \gg R, \quad r_1 \simeq R$, when the particles are near the walls

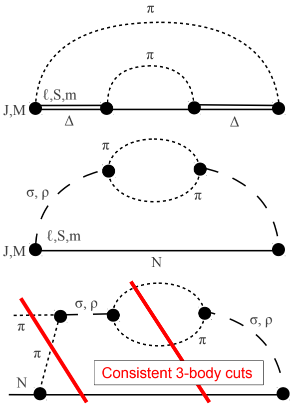
The problem with the disconnected contributions: is the finite-volume spectrum in the 3-particle case determined solely through the on-shell scattering matrix?

- Despite the presence of the disconnected contributions, the energy spectrum of the 3-particle system in a finite box is still determined by the <u>on-shell</u> scattering matrix elements in the infinite volume

[Polejaeva, Rusetski, EPJA (2012)]

Three-particle intermediate states

[See also M. Hansen, S. Sharpe, this conference; here: isobar/dimer picture]

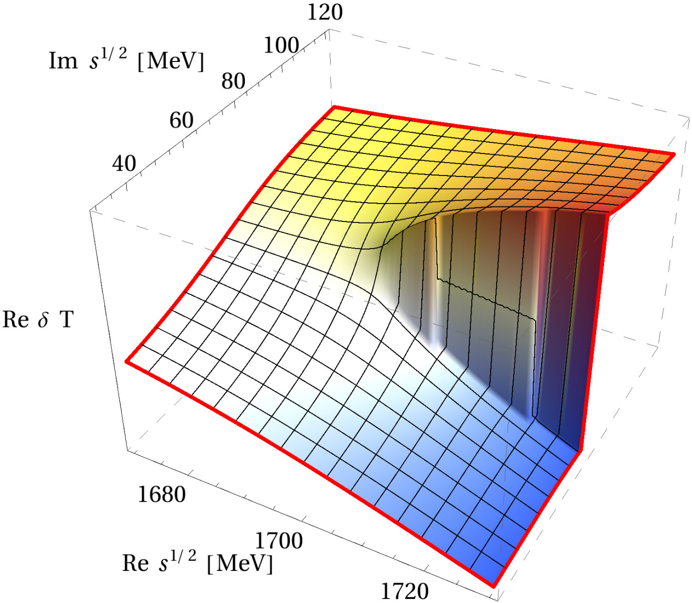


- πN scattering: Known large inelasticities $\pi\pi N$ [$\pi\Delta$, σN, ρN,...]
- $\pi\pi/\pi N$ boosted subsystems.
- Is it enough to include (boosted) 2-particle subsystems in the propagator? **No.**
- Three-body s-channel dynamics requires particle exchange transitions. $\Rightarrow$ **Three-body unitarity**

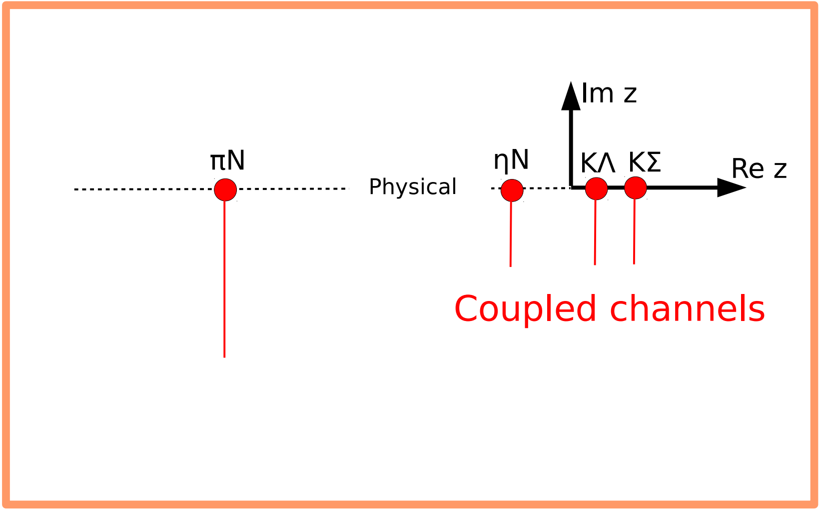
[Aaron, Almado, Young, PR 174 (1968) 2022,
Aitchison, Brehm, PLB 84 (1979) 349, PRD 25 (1982) 3069,...]

Consequence: Threshold openings in the complex plane

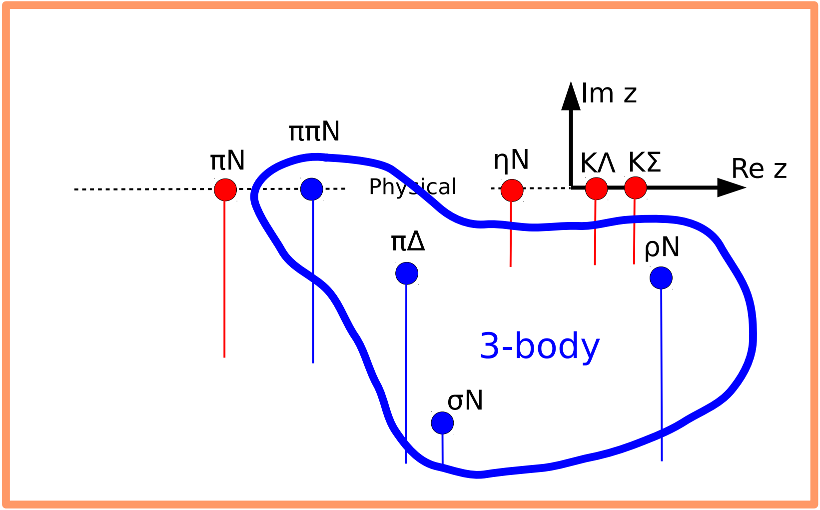
Existence shown model-independently in [S. Ceci, M.D., C. Hanhart *et. al.*, PRC 84 (2011)]



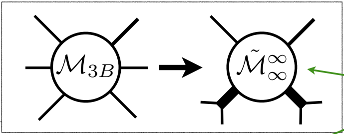
$(z = E)$



$(z = E)$



Three-particle system via dimer formalism

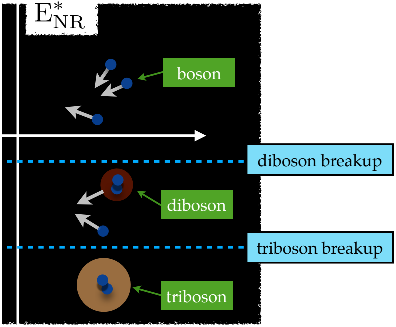


In general, related via integral equation

Quantization condition:

$$\det_{\text{oc}}\left[\det_{\text{pw}}\left(1+\tilde{\mathcal{M}}^{\infty}_{\text{V}}\,\delta\tilde{\mathcal{G}}^{\text{V}}\right)\right]=0$$

oc = open channels, pw = partial waves



Below diboson breakup:

diboson CM binding energy

CM momentum: $q^{*2}_{Bd}\equiv\frac{4m}{3}\left(E^*+\frac{\gamma^{*2}_d}{m}\right)$

Boson-diboson scat. phase shift: δ_{Bd}

$$q^*_0\cot\delta_{Bd}=\frac{2}{L\pi}\mathcal{Z}^{\mathbf{P}}_{00}[1;(q^*_0L/2\pi)^2]+\frac{\eta}{L}e^{-\gamma^*_dL}$$

Extrapolate to infinite volume!

Above breakup: integral equation involving “coupled channels”

Raúl Briceño & Zohreh Davoudi
PRD 87:094507, arXiv:1212.3398[hep-lat]

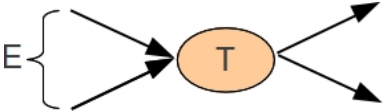
Perspectives

- Close to the physical point, finite volume effects dominate the spectrum.
- Use finite volume effects in your favor: Lüscher & extensions (coupled channels, moving frames, twisted boundary conditions,...)
- Energy interpolation needed in many aspects (Unitarized ChPT & other EFTs provide a framework).
- Rapid progress in the actual ab-initio calculations of resonances/phase shifts: $\rho(770), a_0(980), K^*$(s, p, d)$, N(1535), N(1650), \Delta(1232), \ldots$.

- Thank you to the Organizers!
- Thank you for slides: R. Briceño, G. Engel, C. Lang, B. Menadue, M. Petschlies, A. Rusetsky, G. Schierholz, M. Wagner, D. Wilson.

- Unitarity of the scattering matrix S: $SS^\dagger = \mathbb{1}$ $\quad [S = \mathbb{1} - i \,\frac{p}{4\pi E}\, T]$.

$$\mathrm{Im}\, T^{-1}(E) = \sigma \equiv \frac{p}{8\pi\, E}$$



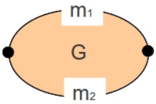
- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the T-matrix:

$$T = V + V\, G\, T \qquad \mathrm{Im}\, G = -\sigma$$

V: (Pseudo)potential, σ: phase space.

- G: Green's function:

$$G = \int \frac{d^3\vec{q}}{(2\pi)^3}\, \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon},$$
$$\omega_{1,2}^2 = m_{1,2}^2 + \vec{q}^{\,2}$$



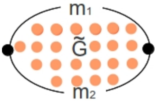
Discretization

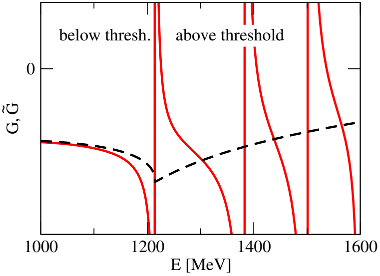
Discretized momenta in the finite volume with periodic boundary conditions

$$\Psi(x) \stackrel{!}{=} \Psi(x + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(x) \quad \Longrightarrow \quad q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \rightarrow \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\, \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

$$G \rightarrow \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}$$





- $E > m_1 + m_2$: $\tilde{G}$ has poles at *free energies in the box*, $E = \omega_1 + \omega_2$
- $E < m_1 + m_2$: $\tilde{G} \rightarrow G$ exponentially with L (regular summation theorem).
- Formalism can be mapped to Lüscher's $\mathcal{Z}_{\ell m}$.

Finite $\rightarrow$ infinite volume: the Lüscher equation

Warning: Very crude re-derivation.

- Measured eigenvalues of the Hamiltonian (tower of *lattice levels* $E(L)$) $\rightarrow$ Poles of scattering equation $\tilde{T}$ in the finite volume $\rightarrow$ determines V:

$$\tilde{T} = (1 - V\tilde{G})^{-1} V \;\rightarrow\; V^{-1} - \tilde{G} \overset{!}{=} 0 \;\rightarrow\; V^{-1} = \tilde{G}$$

- The interaction V determines the T-matrix in the infinite volume limit:

$$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher's equation (T determines the phase shift δ):

$$\boxed{p \cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

- V and dependence on renormalization have disappeared (!)